Let $Q = [0,1]^{\mathbb{N}} = \{ f : \mathbb{N} \to [0,1] \}$.

We give Q the topology of pointwise convergence: a sequence $f_n \in Q$ converges to $f \in Q$ iff $f_n(m) \xrightarrow[n\to\infty]{} f(m)$ $\forall m \in \mathbb{N}$.

Theorem: (Tychonoff) Q is sequentially compact:

Any sequence $\{f_n\}_{n=1}^{\infty}$ in Q has a convergent subsequence.

Pf. The sequence $\{f_n(1)\}_{n=1}^{\infty}$ in $[0,1]$ has a convergent subsequence (because $[0,1]$ is compact); taking a further subsequence if necessary, we can find $n_1(1) < n_1(2) < \cdots < n_1(k) < \cdots$ and $x_1 \in [0,1]$ s.t.

$$| f_{n_1(k)}(1) - x_1 | < 2^{-k}$$

Now consider $\{ f_{n_1(k)}(2) \}_{k=1}^{\infty}$. By the same reasoning, we can find indices $n_2(1) < n_2(2) < \cdots < n_2(k) < \cdots$ selected from among $\{n_1(k)\}_{k=1}^{\infty}$ such that

$$| f_{n_2(k)}(2) - x_2 | < 2^{-k}.$$

By construction, $\{f_{n_2(k)}\}_{k=1}^{\infty}$ is a subsequence of $\{f_{n_1(k)}\}_{k=1}^{\infty}$, and so

$$|f_{n_2(k)}(1) - x_1| \leq 2^{-k}$$

as well. Proceeding inductively, we constructed nested increasing sequences of indices $\{n_1(k)\}_{k=1}^{\infty} \supseteq \{n_2(k)\}_{k=1}^{\infty} \supseteq \cdots \supseteq \{n_m(k)\}_{k=1}^{\infty} \supseteq \cdots$ and $x_j \in [0,1]$ s.t.
s.t.

$$|f_{n_m(k)}(j) - x_j| < 2^{-k} \quad \forall j \leq m, \forall k. \qquad (\star)$$

Claim: $\{g_k\}_{k=1}^{\infty} := \{f_{n_k(k)}\}_{k=1}^{\infty}$ is a subsequence of $\{f_n\}_{n=1}^{\infty}$ and $g_k(j) \xrightarrow[k\to\infty]{} x_j \ \forall j$.

Because $\{n_2(k)\}_{k=1}^{\infty} \subseteq \{n_1(k)\}_{k=1}^{\infty}$ and both $n_1, n_2 \uparrow$, $n_2(1) \geq n_1(1)$; but $n_2(2) > n_2(1)$ because $n_2 \uparrow$, $\therefore n_2(2) > n_1(1)$. Repeating this argument shows that $n_1(1) < n_2(2) < \cdots < n_k(k) < \cdots$ and so $\{f_{n_k(k)}\}_{k=1}^{\infty}$ is a subsequence.

Now, by $(\star)$ with $m=k$, $|g_k(j) - x_j| = |f_{n_k(k)}(j) - x_j| < 2^{-k} \ \forall j \leq k$.
Holding j fixed, it follows that $g_k(j) \to x_j$ as $k \to \infty$.
Thus, $\{g_k\}_{k=1}^{\infty}$ is a subsequence that converges pointwise to $g(j) = x_j$.
$\leftarrow g \in Q$.

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