Regular Borel Measures

If Ω is a (locally compact Hausdorff) topological space, a measure μ on $\mathcal{B}(\Omega)$ is called

- **outer-regular** if $\mu(B) = \inf\{\mu(V) : B \subseteq V,\ V \text{ open}\}$
- **inner-regular** if $\mu(B) = \sup\{\mu(K) : K \subseteq B,\ K \text{ compact}\}$

/ We showed (Lecture 4.2) that the outer measure of a Borel premeasure is outer regular on $\mathcal{B}(\Omega)$ — the Lebesgue σ-field. /

Definition: A Borel measure μ is a **Radon measure** if it is locally finite: $\mu(K) < \infty$ $\forall K \subseteq \Omega$ compact, and it is both outer- and inner-regular.

Theorem: [13.17] All finite (e.g. probability) Borel measures on $\mathbb{R}^d$ are Radon measures.

Pf. Define $\mathcal{F} := \{B \in \mathcal{B}(\mathbb{R}^d) : \forall \varepsilon > 0 \ \exists$ open V, closed C s.t. $C \subseteq B \subseteq V, \mu(V \setminus C) < \varepsilon\}$.

We will show that $\mathcal{F} = \mathcal{B}(\mathbb{R}^d)$. This suffices:

$$\begin{array}{ccc} C \subseteq B \subseteq V & \Rightarrow & V \setminus B \subseteq V \setminus C \\ \Downarrow & & \\ B \setminus C \subseteq V \setminus C & & \end{array}$$

We will show that $\mathcal{F}$ is a σ-field containing all closed sets.

1. $\boxed{\mathcal{F} \text{ contains all closed sets:}}$ Let C be closed. Fix $\varepsilon > 0$, let $C_\varepsilon = \bigcup_{x \in C} B(x, \varepsilon)$.

2. $\boxed{\mathcal{F} \text{ is an algebra.}}$ · $\emptyset \in \mathcal{F}$

· If $A \in \mathcal{F}$, find $C \subseteq A \subseteq V$ with $\mu(V \setminus C) < \varepsilon$.

· If $A_1, A_2 \in \mathcal{F}$, find $C_j \subseteq A_j \subseteq V_j$ with $\mu(V_j \setminus C_j) < \varepsilon/2$.

3. $\boxed{\mathcal{F} \text{ is closed under countable disjoint union.}}$ $A_n \in \mathcal{F}$, find $C_n \subseteq A_n \subseteq V_n$

disjoint with $\mu(V_n \setminus C_n) < \varepsilon/2^{n+1}$.

Fix $N \in \mathbb{N}$, let $D_N = C_1 \cup \cdots \cup C_N$

$V = \bigcup_{n=1}^{\infty} V_n$

$\Big\}$ $D_N \subseteq \bigsqcup_{n=1}^{\infty} A_n \subseteq V$

Recall the ∞-cube $Q = [0,1]^{\mathbb{N}}$, equipped with the topology of pointwise convergence.
We showed that Q is compact, & therefore has the finite intersection property.

Theorem: (Kolmogorov)
Let ν_n be a probability measure on $([0,1]^n, \mathcal{B}([0,1]^n))$, and suppose these measures satisfy the following consistency condition:

$$\nu_{n+1}(B \times [0,1]) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n)$$

Then there exists a unique probability measure $\mathbb{P}$ on $(Q, \mathcal{B}(Q))$ s.t.

$$\mathbb{P}(B \times Q) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n).$$

Pf. Set $\mathcal{B}_n = \{B \times Q : B \in \mathcal{B}([0,1]^n)\}$

$= \sigma\{\pi_1, \dots, \pi_n\}$ where $\pi_k : Q \to [0,1]$

$\pi_k((x_n)_{n=1}^{\infty}) = x_k$.

Let $\mathcal{A} := \bigcup_{n \geq 1} \mathcal{B}_n$. Thus $\mathcal{A}$ is an algebra. Also, if $C \subseteq Q$ is closed, let $B_n = \pi_1 \times \cdots \times \pi_n(C) \subseteq [0,1]^n$, closed.

Then $C = \bigcap_{n \geq 1}^{\infty} (\pi_1 \times \cdots \times \pi_n)^{-1}(B_n) \Rightarrow C \in \sigma\{\pi_n : n \in \mathbb{N}\} = \sigma(\mathcal{A})$.

$\Rightarrow \mathcal{B}(Q) = \sigma(\mathcal{A})$.

Now, define: $\boxed{P(A \times Q) := \nu_n(A) \quad \forall A \in \mathcal{A}.}$ $(\star)$

Using the consistency condition, we see that P is a finitely-additive measure on $\mathcal{A}$. [HW]

Thus, it suffices to show that P is a premeasure on $\mathcal{A}$.

Then it extends to a measure $\bar{P}$ on $\bar{\mathcal{A}}$. Set $\mathbb{P} := \bar{P}|_{\sigma(\mathcal{A}) = \mathcal{B}(Q)}$.

Then $(\star)$ will hold for all $A \in \sigma(\mathcal{A})$ (e.g. by the MCT).

Thus, suffices to show $P(A_n) \downarrow 0$ whenever $A_n \downarrow \emptyset$, $A_n \in \mathcal{A}$.

I.e.: we will show that, if $B_n \in \mathcal{A}$, $B_n \downarrow$, and $\inf_n P(B_n) = \varepsilon > 0$,

then $B := \bigcap_n B_n \neq \emptyset$.

Claim: Suffices to assume $B_n \in \mathcal{B}_n$.

So, $B_n \in \mathcal{B}_n$, $\therefore$ $B_n = B_n' \times Q$. By regularity, find compact $K_n' \subseteq B_n'$ s.t.

$$\nu_n(B_n' \setminus K_n') < \varepsilon/2^{n+1}$$

$$\therefore K_n := K_n' \times Q \Rightarrow P(B_n \setminus K_n) < \varepsilon/2^{n+1}$$

$\therefore$ if $B_n \in \mathcal{B}_n$, $\exists\, K_n \in \mathcal{B}_n$ $K_n = K_n' \times Q$ s.t. $P(B_n \setminus K_n) < \frac{\varepsilon}{2^{n+1}}$.

Thus, $P\left(B_n \setminus \bigcap_{i=1}^{n} K_i\right) = P\left(\bigcup_{i=1}^{n} (B_n \setminus K_i)\right) \leq \sum_{i=1}^{n} P(B_n \setminus K_i) \leq \sum_{i=1}^{n} P(B_i \setminus K_i) < \sum_{i=1}^{n} \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}$.

But we assumed $\inf_n P(B_n) = \varepsilon > 0$. Thus

$P\left(\bigcap_{i=1}^{n} K_i\right)$

In particular, we conclude that $\bigcap_{i=1}^{n} K_i$

Cor: Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ the projection $\pi_n((x_k)_{k=1}^{\infty}) = x_n$

$\mathcal{B}_n := \sigma\{\pi_k : k \leq n\}$, $\mathcal{B} := \sigma(\mathcal{B}_n : n \in \mathbb{N})$

Let ν_n be Borel probability measures on $\mathbb{R}^n$ s.t.

$$\nu_{n+1}(B \times \mathbb{R}) = \nu_n(B) \qquad \forall B \in \mathcal{B}(\mathbb{R}^n)$$

Then $\exists!$ probability measure $\mathbb{P}$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B})$ s.t.

$$\mathbb{P}(B \times \mathbb{R}^{\mathbb{N}}) = \nu_n(B) \qquad \forall B \in \mathcal{B}(\mathbb{R}^n).$$

Pf. Fix a homeomorphism $\alpha : \mathbb{R} \to (0,1)$

$\therefore$ $\alpha^n : \mathbb{R}^n \to (0,1)^n$ is a homeomorphism.

Thus $B \in \mathcal{B}(\mathbb{R}^n) \iff A = \alpha^n(B) \in \mathcal{B}((0,1)^n) \subseteq \mathcal{B}([0,1]^n)$.

Set $\tilde{\nu}_n(A) :=$

$\tilde{\nu}_{n+1}(A \times [0,1]) =$

Cor: Let μ_n be Borel probability measures on $\mathbb{R}$. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of independent random variables, s.t.

$$\mu_{X_n} = \mu_n \quad \forall n \in \mathbb{N}.$$

Pf. Take $\Omega = \mathbb{R}^{\mathbb{N}}$, $\mathcal{F} = \sigma\{\pi_n : n \in \mathbb{N}\}$. Define $\nu_n = \mu_1 \otimes \cdots \otimes \mu_n$. Then

$$\nu_{n+1}(B \times \mathbb{R})$$

$\therefore$ By Kolmogorov, $\exists\, \mathbb{P} \in \mathrm{Prob}(\mathcal{F})$ s.t. $\mathbb{P}(B \times \mathbb{R}^{\mathbb{N}}) = \nu_n(B) \ \forall B \in \mathcal{B}(\mathbb{R}^n)$

Claim: $X_n =$