Regular Borel Measures

If Ω is a (locally compact Hausdorff) topological space, a measure μ on $\mathcal{B}(\Omega)$ is called

- outer-regular if $\mu(B) = \inf\{\mu(V) : B \subseteq V, V \text{ open}\}$
- inner-regular if $\mu(B) = \sup\{\mu(K) : K \subseteq B, K \text{ compact}\}$

/ We showed (Lecture 4.2) that the outer measure of a Borel premeasure is outer regular on $\mathcal{B}(\Omega)$ — the Lebesgue σ-field. /

Re-Definition: A Borel measure μ is a Radon measure if it is locally finite: $\mu(K) < \infty \ \forall K \subseteq \Omega$ compact, and it is both outer- and inner-regular.

Theorem: [13.17] All finite (e.g. probability) Borel measures on $\mathbb{R}^d$ are Radon measures.

Pf. Define $\mathcal{F} := \{B \in \mathcal{B}(\mathbb{R}^d) : \forall \varepsilon > 0 \ \exists$ open V, closed C s.t. $C \subseteq B \subseteq V$, $\mu(V \setminus C) < \varepsilon\}$.

We will show that $\mathcal{F} = \mathcal{B}(\mathbb{R}^d)$. This suffices:

$C \subseteq B \subseteq V \Rightarrow V \setminus B \subseteq V \setminus C$ $\therefore$ we can find $V_n \supseteq B$ s.t. $\underbrace{\mu(V_n \setminus B)}_{= \mu(V_n) - \mu(B)} < \frac{1}{n}$

$\Downarrow$

$B \setminus C \subseteq V \setminus C$

$\therefore$ can find closed C_n s.t. $\mu(B) - \mu(C_n) < \frac{1}{n}$

$\mu(B) = \sup\{\mu(C) : C \subseteq B,\ C \text{ closed}\}$

Also, $\overline{\mathbb{B}}^d(0,n) \uparrow_{n\to\infty} \mathbb{R}^d$, $\therefore$ $\underbrace{\overline{\mathbb{B}}^d(0,n) \cap C}_{\text{compact}} \uparrow C$ $\therefore$ $\mu(\overline{\mathbb{B}}^d(0,n) \cap C) \uparrow \mu(C)$

We will show that $\mathcal{F}$ is a σ-field containing all closed sets. ✓

1. $\boxed{\mathcal{F} \text{ contains all closed sets:}}$ Let C be closed. Fix $\varepsilon > 0$, let $C_\varepsilon = \bigcup_{x \in C} B(x, \varepsilon)$.

C_ε is open, $C_\varepsilon \downarrow C$ as $\varepsilon \downarrow 0$. $\mu(C_\varepsilon \setminus C) \downarrow 0$.

(in general, $C_\varepsilon \downarrow \bar{C}$) $C \subseteq \bar{C} \subseteq \underbrace{C_\varepsilon}_{\text{open}}$

2. $\boxed{\mathcal{F} \text{ is an algebra.}}$ ✓. $\emptyset \in \mathcal{F}$ $\quad \emptyset \subseteq \emptyset \subseteq \emptyset \quad \mu(\emptyset \setminus \emptyset) = 0.$

· If $A \in \mathcal{F}$, find $C \subseteq A \subseteq V$ with $\mu(V \setminus C) < \varepsilon$.

$$\underset{\text{closed}}{V^c} \subseteq A^c \subseteq \underset{\text{open}}{C^c} \qquad C^c \setminus V^c = C^c \cap (V^c)^c = C^c \cap V = V \setminus C$$

$$\therefore \mu(C^c \setminus V^c) = \mu(V \setminus C) < \varepsilon.$$

· If $A_1, A_2 \in \mathcal{F}$, find $C_j \subseteq A_j \subseteq V_j$ with $\mu(V_j \setminus C_j) < \varepsilon/2$.

$$C := C_1 \cup C_2 \subseteq A_1 \cup A_2 \subseteq V_1 \cup V_2 =: V.$$

$$\mu(V \setminus C) = \mu((V_1 \setminus C) \cup (V_2 \setminus C)) \leq \mu(V_1 \setminus C) + \mu(V_2 \setminus C)$$

$$\leq \mu(V_1 \setminus C_1) + \mu(V_2 \setminus C_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

3. $\boxed{\mathcal{F} \text{ is closed under countable disjoint union.}}$ ✓ $\underset{\text{disjoint}}{A_n} \in \mathcal{F}$, find $C_n \subseteq A_n \subseteq V_n$ with $\mu(V_n \setminus C_n) < \varepsilon/2^{n+1}$.

Fix $N \in \mathbb{N}$, let $D_N = C_1 \cup \cdots \cup C_N \leftarrow$ closed

$V = \bigcup_{n=1}^{\infty} V_n \leftarrow$ open

$\Big\}\ D_N \subseteq \bigsqcup_{n=1}^{\infty} A_n \subseteq V$

$$\mu(V \setminus D_N) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus D_N) \leq \sum_{n=1}^{N} \underbrace{\mu(V_n \setminus C_n)}_{< \varepsilon/2^{n+1}} + \sum_{n=N+1}^{\infty} \underbrace{\mu(V_n)}_{< \mu(A_n) + \frac{\varepsilon}{2^{n+1}}} \leq \frac{\varepsilon}{2} + \underbrace{\sum_{n=N+1}^{\infty} \mu(A_n)}_{\mu\left(\bigsqcup_{n=N+1}^{\infty} A_n\right)} \qquad ///$$

Recall the ∞-cube $Q = [0,1]^{\mathbb{N}}$, equipped with the topology of pointwise convergence. We showed that Q is compact, & therefore has the finite intersection property.

Theorem: (Kolmogorov)

Let ν_n be a probability measure on $([0,1]^n, \mathcal{B}([0,1]^n))$, and suppose these measures satisfy the following consistency condition:

$$\nu_{n+1}(B \times [0,1]) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n)$$

Then there exists a unique probability measure $\mathbb{P}$ on $(Q, \mathcal{B}(Q))$ s.t.

$$\mathbb{P}(B \times Q) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n).$$

Important special case:

$$\nu_n = \mu_1 \otimes \cdots \otimes \mu_n \qquad \mu_j \text{ a Borel prob. meas. on } [0,1].$$

$$\nu_{n+1} = \mu_1 \otimes \cdots \otimes \mu_n \otimes \mu_{n+1} = \nu_n \otimes \mu_{n+1}$$

$$\nu_{n+1}(B \times [0,1]) = \nu_n \otimes \mu_{n+1}(B \times [0,1])$$

$$= \nu_n(B)\underbrace{\mu_{n+1}([0,1])}_{1} = \nu_n(B).$$

Pf. Set $\mathcal{B}_n = \{B \times Q : B \in \mathcal{B}([0,1]^n)\}$

$= \sigma\{\pi_1, \dots, \pi_n\}$ where $\pi_k : Q \to [0,1]$

$\pi_k((x_n)_{n=1}^{\infty}) = x_k$.

Let $\mathcal{A} := \bigcup_{n \geq 1} \mathcal{B}_n$. Thus $\mathcal{A}$ is an algebra. Also, if $C \subseteq Q$ is closed, let $B_n = \pi_1 \times \cdots \times \pi_n(C) \subseteq [0,1]^n$, closed.

Then $C = \bigcap_{n \geq 1}^{\infty} (\pi_1 \times \cdots \times \pi_n)^{-1}(B_n) \Rightarrow C \in \sigma\{\pi_n : n \in \mathbb{N}\} = \sigma(\mathcal{A})$.

$$\Rightarrow \mathcal{B}(Q) = \sigma(\mathcal{A}).$$

Now, define: $\boxed{P(A \times Q) := \nu_n(A) \quad \forall A \in \mathcal{A}.}$ $(\star)$

Using the consistency condition, we see that P is a finitely-additive measure on $\mathcal{A}$. [HW]

Thus, it suffices to show that P is a premeasure on $\mathcal{A}$. Then it extends to a measure $\bar{P}$ on $\bar{\mathcal{A}}$. Set $\mathbb{P} := \bar{P}|_{\sigma(\mathcal{A}) = \mathcal{B}(Q)}$. Then $(\star)$ will hold for all $A \in \sigma(\mathcal{A})$ (e.g. by the MCT).

Thus, suffices to show $P(A_n) \downarrow 0$ whenever $A_n \downarrow \emptyset$, $A_n \in \mathcal{A}$.

I.e.: we will show that, if $B_n \in \mathcal{A}$, $B_n \downarrow$, and $\inf_n P(B_n) = \varepsilon > 0$,

then $B := \bigcap_n B_n \neq \emptyset$.

Claim: Suffices to assume $B_n \in \mathcal{B}_n$.

$B_n \in \mathcal{A} = \bigcup_n \mathcal{B}_n \Rightarrow B_n \in \mathcal{B}_{m_n}$

$$\begin{array}{cccccccccccccccccc} & 1 & 2 & \cdots & & m_1 & & \cdots & & m_2 & & \cdots & & m_3 & \cdots & n \\ (\tilde{B}_k) = (& Q, & Q, & \cdots & Q, & B_1, & B_1, & \cdots, & B_1, & B_2, & B_2, & \cdots & B_2, & B_3, & \cdots &) \\ & \tilde{B}_1 & \tilde{B}_2 & & \tilde{B}_{m_1-1} & \tilde{B}_{m_1} & & & & \tilde{B}_{m_2} & & & & & & \end{array}$$

$\therefore \tilde{B}_k \in \mathcal{B}_k$. $\tilde{B}_k \downarrow$

$\inf_k P(\tilde{B}_k) = \inf_n P(B_n) = \varepsilon$

$\bigcap_k \tilde{B}_k = \bigcap_n B_n$.

So, $B_n \in \mathcal{B}_n$, $\therefore B_n = B_n' \times Q$. By regularity, find compact $K_n' \subseteq B_n'$ s.t.

$B_n' \in \mathcal{B}([0,1]^n)$

$\nu_n(B_n' \setminus K_n') < \varepsilon / 2^{n+1}$

$\therefore K_n := K_n' \times Q \Rightarrow P(B_n \setminus K_n) < \varepsilon / 2^{n+1}$

$\therefore$ if $B_n \in \mathcal{B}_n$, $\exists\, K_n \in \mathcal{B}_n$ $K_n = K_n' \times Q$ s.t. $P(B_n \setminus K_n) < \frac{\varepsilon}{2^{n+1}}$.

Thus, $P(B_n \setminus \bigcap_{i=1}^{n} K_i) = P(\bigcup_{i=1}^{n} (B_n \setminus K_i)) \leq \sum_{i=1}^{n} P(B_n \setminus K_i) \underset{B_n \downarrow}{\leq} \sum_{i=1}^{n} P(B_i \setminus K_i) < \sum_{i=1}^{n} \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}$.

But we assumed $\inf_n P(B_n) = \varepsilon > 0$. Thus

$P(\bigcap_{i=1}^{n} K_i) \longrightarrow P(B_n) - P(B_n \setminus \bigcap_{i=1}^{n} K_i) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0$.

$B_n \setminus (B_n \setminus \bigcap_{i=1}^{n} K_i)$

In particular, we conclude that $\bigcap_{i=1}^{n} K_i \neq \emptyset$. $\forall n$.

$\therefore \bigcap_{i=1}^{\infty} K_i \neq \emptyset$

$\bigcap_{i=1}^{\infty} B_i$

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Cor: Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ the projection $\pi_n((x_k)_{k=1}^{\infty}) = x_n$

$\mathcal{B}_n := \sigma\{\pi_k : k \le n\}$, $\mathcal{B} := \sigma(\mathcal{B}_n : n \in \mathbb{N})$

Let ν_n be Borel probability measures on $\mathbb{R}^n$ s.t.

$$\nu_{n+1}(B \times \mathbb{R}) = \nu_n(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^n)$$

Then $\exists!$ probability measure $\mathbb{P}$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B})$ s.t.

$$\mathbb{P}(B \times \mathbb{R}^{\mathbb{N}}) = \nu_n(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^n).$$

Pf. Fix a homeomorphism $\alpha : \mathbb{R} \to (0,1)$ $\left(\alpha(x) = \int_{-\infty}^{x} f(t)\,dt\right.$ $f > 0$ conts. prob. density. e.g. $\left. f(t) = \frac{1}{\pi} \cdot \frac{1}{1+t^2}\right)$

$\therefore$ $\alpha^n : \mathbb{R}^n \to (0,1)^n$ is a homeomorphism.

Thus $B \in \mathcal{B}(\mathbb{R}^n) \iff A = \alpha^n(B) \in \mathcal{B}((0,1)^n) \subseteq \mathcal{B}([0,1]^n)$.

Set $\tilde{\nu}_n(A) := (\alpha^n)^{\#}\nu_n(A \cap (0,1)^n) \leftarrow$ Borel prob. meas. on $[0,1]^n$.

$\tilde{\nu}_{n+1}(A \times [0,1]) = \tilde{\nu}_n(A) \quad \forall A \in \mathcal{B}([0,1]^n)$.

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Cor: Let μ_n be Borel probability measures on $\mathbb{R}$. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of independent random variables, s.t.

$$\mu_{X_n} = \mu_n \quad \forall n \in \mathbb{N}.$$

Pf. Take $\Omega = \mathbb{R}^{\mathbb{N}}$, $\mathcal{F} = \sigma\{\pi_n : n \in \mathbb{N}\} = \bigotimes_{n=1}^{\infty} \underbrace{\mathcal{B}_n}_{\sigma(\pi_1, \ldots, \pi_n)}$. Define $\nu_n = \mu_1 \otimes \cdots \otimes \mu_n$. Then

$$\nu_{n+1}(B \times \mathbb{R}) = \nu_n(B). \quad \checkmark$$

$\therefore$ By Kolmogorov, $\exists\, \mathbb{P} \in \mathrm{Prob}(\mathcal{F})$ s.t. $\mathbb{P}(B \times \mathbb{R}^{\mathbb{N}}) = \nu_n(B)$ $\forall B \in \mathcal{B}(\mathbb{R}^n)$

Claim: $X_n = \pi_n$ does the trick. Fix $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned}
\mathbb{P}(\pi_1 \in B_1, \ldots, \pi_n \in B_n) &= \mathbb{P}(\pi_1^{-1}(B_1) \cap \cdots \cap \pi_n^{-1}(B_n)) = \mathbb{P}(B_1 \times \cdots \times B_n \times \mathbb{R}^{\mathbb{N}}) \\
&= \nu_n(B_1 \times \cdots \times B_n) \\
&= \mu_1(B_1) \cdots \mu_n(B_n). \\
&= \mu_{\pi_1}(B_1) \cdots \mu_{\pi_n}(B_n) \quad ///
\end{aligned}$$