iid Random Variables

A sequence $\{X_n\}_{n=1}^{\infty}$ of random variables

$$X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B})$$

is called iid = independent and identically distributed
if all the X_n are independent, and $\mu_{X_n} = \mu_{X_1}$ $\forall n \in \mathbb{N}$.

But how do we know such things exist?

In general, we would like to construct sequences $\{X_n\}_{n=1}^{\infty}$ of independent random variables/vectors with any prescribed laws: $\{\mu_n\}_{n=1}^{\infty}$ on $(S, \mathcal{B})$

$$\mu_{X_n} = \mu_n$$

For finite sequences, this is easy, and instructive.

Lemma: Let $\mu_1, \dots, \mu_n$ be probability measures on $(S_1, \mathcal{B}_1), \dots, (S_N, \mathcal{B}_N)$. Define

$$\Omega =$$

$$\mathcal{F} =$$

$$\mathbb{P} =$$

Then the random variables $X_n : \Omega \to S_n$

are indendent, and $\mu_{X_n} = \mu_n$.

Pf.

Eg. To construct d iid $\mathcal{N}(0,1)$ random variables,

set $\gamma(x) = (2\pi)^{-1/2} e^{-x^2/2}$, and $d\mu = \gamma \, d\lambda$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Then equip $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with $\mathbb{P} = \mu^{\otimes d}$.

$\therefore \underline{X} = (X_1, \dots, X_d)$ with $X_n(x_1, \dots, x_d) = x_n$ are iid. $\mathcal{N}(0,1)$.

Since μ_{X_j} has a density γ wrt λ,

$\Rightarrow \mathbb{P} = \mu^{\otimes d}$ has density $\gamma \otimes \cdots \otimes \gamma$

[HW]

Kolmogorov's Extension Theorem

We'd like to construct i.i.d. sequences by taking products. That means we need to be able to take infinite products of probability spaces.

Setup. Want a probability measure on (say)

$$\mathbb{R}^{\mathbb{N}} = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R} \ \forall n \in \mathbb{N}\}$$

To take advantage of compactness results, we replace $\mathbb{R}$ with $[0,1]$.

$$Q := [0,1]^{\mathbb{N}}.$$ ← we give it a topology consistent with the above inclusions $[0,1]^d \hookrightarrow Q$.

Def: Q is given the topology of pointwise convergence:

$x^1, x^2, \dots, x^k \in Q$ converge to $x \in Q$ iff

Theorem: (Tychonoff)

Q is (sequentially) compact. I.e.
If $(x^m)_{m=1}^{\infty}$ is a sequence in Q,
it has a convergent subsequence $(x^{m_k})_{k=1}^{\infty}$.

Pf.

Cor: (Finite Intersection Property)
If $K_m \subseteq Q$ are closed subsets s.t. $\bigcap_{i=1}^{m} K_i \neq \phi$ $\forall m \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} K_i \neq \phi$.

Pf.

Theorem: (Kolmogorov)

Let ν_n be a probability measure on $([0,1]^n, \mathcal{B}([0,1]^n))$, and suppose these measures satisfy the following consistency condition:

$$\nu_{n+1}(B \times [0,1]) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n)$$

Then there exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$ s.t.

$$\mathbb{P}(B \times \Omega) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n).$$

Once we prove this, it will generalize almost instantly from $[0,1]$ to $\mathbb{R}$ (and then to $\mathbb{R}^d$).