iid Random Variables

A sequence $\{X_n\}_{n=1}^\infty$ of random variables

$$X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

is called iid = independent and identically distributed

if all the X_n are independent, and $\underset{\substack{\shortparallel \\ X_{n*}\mathbb{P}}}{\mu_{X_n}} = \underset{\substack{\shortparallel \\ X_{1*}\mathbb{P}}}{\mu_{X_1}}$ $\forall n \in \mathbb{N}$.

But how do we know such things exist?

In general, we would like to construct sequences $\{X_n\}_{n=1}^\infty$ of independent random variables / vectors with any prescribed laws: $\{\mu_n\}_{n=1}^\infty$ on $(S, \mathcal{B})$

$$\mu_{X_n} = \mu_n$$

For <u>finite</u> sequences, this is easy, and instructive.

Lemma: Let $\mu_1, \dots, \mu_N$ be probability measures on $(S_1, \mathcal{B}_1), \dots, (S_N, \mathcal{B}_N)$. Define

$$\Omega = S_1 \times \cdots \times S_N$$
$$\mathcal{F} = \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_N$$
$$\mathbb{P} = \mu_1 \otimes \cdots \otimes \mu_N$$

Then the random variables $X_n : \Omega \to S_n$

$$X_n = \pi_n(\underline{x}) = x_n, \quad \underline{x} = (x_1, \dots, x_N)$$

are indendent, and $\mu_{X_n} = \mu_n$.

Pf. $B_n \in \mathcal{B}_n$

$$\mathbb{P}(X_1 \in B_1, \dots, X_N \in B_N) = \mu_1 \otimes \cdots \otimes \mu_N \left(\underline{x} \in S_1 \times \cdots \times S_N : x_1 \in B_1, \dots, x_N \in B_N\right)$$
$$= \mu_1 \otimes \cdots \otimes \mu_N (B_1 \times \cdots \times B_N)$$
$$= \mu_1(B_1) \cdots \mu_N(B_N)$$

apply w/ $B_j = S_j \ \forall j \neq n$

$$\mathbb{P}(X_n \in B_n) = \underbrace{\mu_1(S_1)}_{1} \cdots \mu_n(B_n) \cdots \underbrace{\mu_N(S_N)}_{1} = \mu_n(B_n)$$

Eg. To construct d iid $\mathcal{N}(0,1)$ random variables,
set $\gamma(x) = (2\pi)^{-1/2} e^{-x^2/2}$, and $d\mu = \gamma \, d\lambda$
on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Then equip $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with $\mathbb{P} = \mu^{\otimes d}$.

$\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R})^{\otimes d}$

$\therefore \underline{X} = (X_1, \ldots, X_d)$ with $X_n(x_1, \ldots, x_d) = x_n$ are iid. $\mathcal{N}(0,1)$.

Since μ_{X_j} has a density γ wrt λ,

$\Rightarrow \mathbb{P} = \mu^{\otimes d}$ has density $\gamma \otimes \cdots \otimes \gamma \, (x_1, \ldots, x_d) = (2\pi)^{-1/2} e^{-x_1^2/2} \cdots (2\pi)^{-1/2} e^{-x_d^2/2}$

[HW]

wrt $\lambda^{\otimes d} = \lambda^d$

Lebesgue on $\mathbb{R}^d$

$$= (2\pi)^{-d/2} e^{-\frac{1}{2}\sum_{j=1}^{d} x_j^2}$$

$$= (2\pi)^{-d/2} e^{-\frac{1}{2}\|\underline{x}\|^2}$$

$\mathcal{N}^d(\underline{0}, I_d)$

Kolmogorov's Extension Theorem

we'd like to construct i.i.d. sequences by taking products. That means we need to be able to take infinite products of probability spaces.

Setup. Want a probability measure on (say)

$$\mathbb{R}^{\mathbb{N}} = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R} \ \forall n \in \mathbb{N}\}$$

$$\text{"="} \lim_{d\to\infty} \mathbb{R}^d$$

$$\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \cdots \subset \mathbb{R}^{\mathbb{N}}.$$

$$[0,1]^d \hookrightarrow [0,1]^{\mathbb{N}}$$

$$\mathbb{R}^d \hookrightarrow \mathbb{R}^{\mathbb{N}}$$

$$(x_1, \dots, x_d) \longmapsto (x_1, \dots, x_d, 0, 0, \dots)$$

To take advantage of compactness results, we replace $\mathbb{R}$ with $[0,1]$.

$$Q := [0,1]^{\mathbb{N}}.$$

$\leftarrow$ we give it a topology consistent with the above inclusions $[0,1]^d \hookrightarrow Q$.

Def: Q is given the topology of pointwise convergence:

$x^1 = (x^1_n)_{n=1}^{\infty}, x^2, \dots, x^k \in Q$ converge to $x \in Q$ iff

$$x^k_n \to x_n \quad \forall n \in \mathbb{N}.$$

Theorem: (Tychonoff)

Q is (sequentially) compact. I.e.

If $(x^m)_{m=1}^{\infty}$ is a sequence in Q,

it has a convergent subsequence $(x^{m_k})_{k=1}^{\infty}$.

Pf. $x_1^m \in [0,1] \leftarrow$ compact has a conv. subseq. $x_1^{m_1(k)} \xrightarrow{k} x_1 \in [0,1]$

$x_2^{m_1(k)} \in [0,1]$ " " $x_2^{m_2(k)} \xrightarrow{k} x_2 \in [0,1]$

$\vdots$

$x_j^{m_j(k)} \to x_j \in [0,1]$ Take $x_k^{m_k(k)} \to x_k \quad \forall k$

$x_j^{m_i(k)} \to x_j \quad \forall i \geq j$. (an $\varepsilon/2$-type argument). ///

Cor: (Finite Intersection Property)

If $K_m \subseteq Q$ are closed subsets s.t. $\bigcap_{i=1}^{m} K_i \neq \emptyset \quad \forall m \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$.

Pf. Let $x^m \in \bigcap_{i=1}^{m} K_i$. By Tychonoff, $\exists$ conv. subseq. $x^{m_k} \to x \in Q$.

$x^{m_\ell} \in \bigcap_{i=1}^{m_k} K_i \quad \forall \ell \geq k$. $\therefore x = \lim_{\ell \to \infty} x^{m_\ell} \in K_i \quad \forall i \geq k$. $\therefore x \in \bigcap_{i=1}^{\infty} K_i$ ///

Theorem: (Kolmogorov)

Let ν_n be a probability measure on $([0,1]^n, \mathcal{B}([0,1]^n))$, and suppose these measures satisfy the following consistency condition:

$$\nu_{n+1}(B \times [0,1]) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n)$$

Then there exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$ s.t.

$$\mathbb{P}(B \times \Omega) = \nu_n(B) \qquad \forall B \in \mathcal{B}([0,1]^n).$$

Once we prove this, it will generalize almost instantly from $[0,1]$ to $\mathbb{R}$ (and then to $\mathbb{R}^d$).

$$\mathbb{R} \cong (0,1) \in \mathcal{B}[0,1]$$