Independent Random Variables

$$X_i : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (S_i, \mathcal{B}_i)$$

$\sigma(X_i)$ = minimal σ-field $\subseteq \mathcal{F}$ s.t. X_i is $\mathcal{F}/\mathcal{B}_i$-measurable

Def: Random variables $\{X_i\}_{i\in I}$ are **independent** if the σ-fields $\{\sigma(X_i)\}_{i\in I}$ are independent.

Lemma: Given random variables $X_i : (\Omega, \mathcal{F}, P) \to (S_i, \mathcal{B}_i)$, if $\mathcal{E}_i \subseteq \mathcal{B}_i$ are π-systems s.t. $\sigma(\mathcal{E}_i) = \mathcal{B}_i$, then $\{X_i\}_{i \in I}$ are independent iff

$\{X_i^{-1}(E_i)\}_{i \in I}$ are independent $\forall E_i \in \mathcal{E}_i$

Pf.

Eg. $(S_i, \mathcal{B}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Take $\mathcal{E}_i = \{(-\infty, t] : t \in \mathbb{R}\}$

Thus, $\mathbb{R}$-valued Borel r.v.'s X_i are independent iff $\{X_i^{-1}(-\infty, t_i]\}$ are independent $\forall t_i \in \mathbb{R}$

$$\mathbb{P}(X_1 \leq t_1, \ldots, X_n \leq t_n) = \mathbb{P}(X_1 \leq t_1) \cdots \mathbb{P}(X_n \leq t_n)$$

Given $\underline{X} = (X_1, \dots, X_n)$, $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (S_i, \mathcal{B}_i)$

their joint law $\mu_{\underline{X}}$ is the probability measure on $\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_n$ defined by $\mu_{\underline{X}} := \mathbb{P} \circ \underline{X}^{-1}$

I.e. $\mu_{\underline{X}}(B) =$

Theorem: $X_1, \dots, X_n$ are independent iff

Pf. Let $B_i \in \mathcal{B}_i$, $i \in [n]$. Then $\mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n) =$

Cor: If $X, Y: (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are L^2
then X, Y independent $\Rightarrow$ $\mathrm{Cov}(X,Y) = 0$.

Pf. $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

The converse is generally false. (We'll see later this lecture.)

Prop: If $X_1, \dots, X_n \in L^1$ are independent, then $X_1 X_2 \cdots X_n \in L^1$, and

$$\mathbb{E}[X_1 \cdots X_n] = \mathbb{E}[X_1] \cdots \mathbb{E}[X_n].$$

Pf. Let $\underline{X} = (X_1, \dots, X_n)$. By the independence assumption, $\mu_{\underline{X}} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$.
By Tonelli's theorem and the change of variables formula:

$$\mathbb{E}[|X_1 \cdots X_n|]$$

Theorem: Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (S_i, \mathcal{B}_i)$ be rv's, $i \in [n]$.
Set $\mathbb{X} = (X_1, \dots, X_n)$. TFAE:

1. $X_1, \dots, X_n$ are independent.
2. $\mu_{\mathbb{X}} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$
3. $\mathbb{E}[f_1(X_1) \cdots f_n(X_n)] = \mathbb{E}[f_1(X_1)] \cdots \mathbb{E}[f_n(X_n)]$ $(\star)$ $\forall f_i \in B(S_i, \mathcal{B}_i)$.

Moreover, if each $(S_i, \mathcal{B}_i) = (\mathbb{R}^{d_i}, \mathcal{B}(\mathbb{R}^{d_i}))$, we also have the equivalent conditions

4. $(\star)$ holds $\forall f_i \in C_c(\mathbb{R}^{d_i})$
5. $(\star)$ holds $\forall f_i$ of the form $f_i = \mathbb{1}_{(-\infty, t_1] \times \cdots \times (-\infty, t_{d_i}]}$, $t_1, \dots, t_{d_i} \in \mathbb{R}$

Pf. We've already shown $1 \Leftrightarrow 2$. $2 \Rightarrow 3, 4, 5$ follow from c.o.v. + Fubini's theorem, much like the previous proposition. $4, 5 \Rightarrow 3$ follow from Dynkin's mult. syst. thm.

$3 \Rightarrow 1$:

Groupings and Functions

Lemma: If $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent σ-fields (over Ω), and $n = n_1 + n_2 + \dots + n_k$, then

$$\mathcal{G}_1 = \sigma(\mathcal{F}_1 \cup \dots \cup \mathcal{F}_{n_1}),\ \mathcal{G}_2 = \sigma(\mathcal{F}_{n_1+1} \cup \dots \cup \mathcal{F}_{n_1+n_2}),\ \dots,\ \mathcal{G}_k = \sigma(\mathcal{F}_{n_1+\dots+n_{k-1}+1} \cup \dots \cup \mathcal{F}_n)$$

are independent σ-fields.

Pf.

Cor: Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (S_i, \mathcal{B}_i)$ be independent, $i \in [n]$.

Let $n = n_1 + n_2 + \cdots + n_k$. Let

$f_j : (S_{n_1+\cdots+n_{j-1}+1} \times \cdots \times S_{n_1+\cdots+n_j}, \mathcal{B}_{\circ} \otimes \cdots \otimes \mathcal{B}_{\circ}) \to \mathbb{R}$ be measurable, $j \in [k]$.

Then $Y_j = f_j(X_{n_1+\cdots+n_{j-1}+1}, \ldots, X_{n_1+\cdots+n_j})$ are independent, $j \in [k]$.

Eg. If X_1, X_2, X_3, X_4, X_5 are independent, so are

$$X_1 + X_2, \quad X_3 X_4, \quad e^{X_5}$$

Pf.

Uncorrelated vs. Independent

E.g. $(X,Y) = (X, XZ)$, X, Z independent, $X \in L^2$, $|Z| \leq 1$ with $\mathbb{E}[Z] = 0$.

Then $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

Method of Moments

Proposition: Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be bounded rv's $i \in [n]$

Then $X_1, \dots, X_n$ are independent iff

$$\mathbb{E}[X_1^{k_1} \cdots X_n^{k_n}] = \mathbb{E}[X_1^{k_1}] \cdots \mathbb{E}[X_n^{k_n}], \quad \forall k_1, \dots, k_n \in \mathbb{N}.$$

Pf. $(\Rightarrow)$

$(\Leftarrow)$ [HW].