Independent Random Variables

$$X_i : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (S_i, \mathcal{B}_i) = (\mathbb{R}^{d_i}, \mathcal{B}(\mathbb{R}^{d_i}))$$

$\sigma(X_i)$ = minimal σ-field $\subseteq \mathcal{F}$ s.t. X_i is $\mathcal{F}/\mathcal{B}_i$-measurable

$= X_i^* \mathcal{B}_i = \{ X_i^{-1}(B_i) : B_i \in \mathcal{B}_i \}$

Def: Random variables $\{X_i\}_{i \in I}$ are independent if the σ-fields $\{\sigma(X_i)\}_{i \in I}$ are independent.

I.e. $\forall B_i \in \mathcal{B}_i$, $\{ X_i^{-1}(B_i) \}_{i \in I}$ are independent.

$$\mathbb{P}(X_1^{-1}(B_1) \cap X_2^{-1}(B_2) \cap \cdots \cap X_n^{-1}(B_n)) = \mathbb{P}(X_1^{-1}(B_1))\, \mathbb{P}(X_2^{-1}(B_2)) \cdots \mathbb{P}(X_n^{-1}(B_n))$$

i.e. $\mathbb{P}(X_1 \in B_1, X_2 \in B_2, \dots, X_n \in B_n) = \mathbb{P}(X_1 \in B_1)\, \mathbb{P}(X_2 \in B_2) \cdots \mathbb{P}(X_n \in B_n)$

Lemma: Given random variables $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (S_i, \mathcal{B}_i)$,
if $\mathcal{E}_i \subseteq \mathcal{B}_i$ are π-systems s.t. $\sigma(\mathcal{E}_i) = \mathcal{B}_i$, then
$\{X_i\}_{i\in I}$ are independent iff

$$\{X_i^{-1}(E_i)\}_{i\in I} \text{ are independent } \forall E_i \in \mathcal{E}_i$$

Pf. $\mathcal{C}_i = X_i^* \mathcal{E}_i = \{X_i^{-1}(E_i) : E_i \in \mathcal{E}_i\}$.

- $\mathcal{C}_i$ is a π-system: $A, B \in \mathcal{C}_i$ $\quad A \cap B = X_i^{-1}(E) \cap X_i^{-1}(F) = X_i^{-1}(\underbrace{E \cap F}_{\in \mathcal{E}_i}) \in \mathcal{C}_i$.

  $A = X_i^{-1}(E)$, $B = X_i^{-1}(F)$, $E, F \in \mathcal{E}_i$

- $\sigma(\mathcal{C}_i) = \sigma(X_i^* \mathcal{E}_i) = X_i^*(\sigma(\mathcal{E}_i)) = X_i^* \mathcal{B}_i = \sigma(X_i)$

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Eg. $(S_i, \mathcal{B}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Take $\mathcal{E}_i = \{(-\infty, t] : t \in \mathbb{R}\}$ $\quad (-\infty, t] \cap (-\infty, s] = (-\infty, s \wedge t]$.

Thus, $\mathbb{R}$-valued Borel r.v.'s X_i are independent
iff $\{X_i^{-1}(-\infty, t_i]\}$ are independent $\forall t_i \in \mathbb{R}$

$$\mathbb{P}(X_1 \le t_1, \dots, X_n \le t_n) = \mathbb{P}(X_1 \le t_1) \cdots \mathbb{P}(X_n \le t_n) = F_{X_1}(t_1) \cdots F_{X_n}(t_n)$$

Given $\underline{X} = (X_1, \dots, X_n)$, $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (S_i, \mathcal{B}_i)$

their joint law $\mu_{\underline{X}}$ is the probability measure on $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$ defined by $\mu_{\underline{X}} := \mathbb{P} \circ \underline{X}^{-1}$

$\underbrace{\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n}_{\sigma(B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i),\ \sigma\text{-field over } S_1 \times \cdots \times S_n}$

If $(S_i, \mathcal{B}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n = \mathcal{B}(\mathbb{R}^n)$.

I.e. $\mu_{\underline{X}}(B) = \mathbb{P}(\underline{X} \in B)$.

Theorem: $X_1, \dots, X_n$ are independent iff

$$\mu_{X_1, \dots, X_n} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}.$$

$(\Leftarrow)$ ✓

$(\Rightarrow)$ ✓

Pf. Let $B_i \in \mathcal{B}_i$, $i \in [n]$. Then $\mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n) = \mathbb{P}(\underline{X} \in B_1 \times \cdots \times B_n)$

If $\mu_{\underline{X}} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$

$= \mu_{\underline{X}}(B_1 \times \cdots \times B_n)$

$= \mu_{X_1}(B_1) \cdots \mu_{X_n}(B_n)$

$= \mathbb{P}(X_1 \in B_1) \cdots \mathbb{P}(X_n \in B_n)$

$\therefore X_1, \dots, X_n$ indep ✓.

Conversely, if $X_1, \dots, X_n$ are indep,

$\mathbb{P}(\underline{X} \in B_1 \times \cdots \times B_n) = \mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n)$

$\mu_{\underline{X}}(B_1 \times \cdots \times B_n) = \mathbb{P}(X_1 \in B_1) \cdots \mathbb{P}(X_n \in B_n)$

$= \mu_{X_1}(B_1) \cdots \mu_{X_n}(B_n) = \mu_1 \otimes \cdots \otimes \mu_n(B_1 \times \cdots \times B_n)$.

Cor: If $X, Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are L^2

then X, Y independent $\Rightarrow \mathrm{Cov}(X,Y) = 0$.

Pf. $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

C.o.V. $\mathbb{E}[XY] = \int xy \, \mu_{X,Y}(dx\,dy) = \int xy \, \mu_X(dx)\mu_Y(dy)$

$$= \int \left(\int xy \, \mu_X(dx) \right) \mu_Y(dy)$$

$$= \int y \, \mu_Y(dy) \cdot \int x \, \mu_X(dx)$$

$$= \mathbb{E}[Y]\,\mathbb{E}[X]$$

The converse is generally false. (we'll see later this lecture)

Prop: If $X_1, \dots, X_n \in L^1$ are independent, then $X_1 X_2 \cdots X_n \in L^1$, and

$$\mathbb{E}[X_1 \cdots X_n] = \mathbb{E}[X_1] \cdots \mathbb{E}[X_n].$$

Pf. Let $\mathbb{X} = (X_1, \dots, X_n)$. By the independence assumption, $\mu_{\mathbb{X}} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$.

By Tonelli's theorem and the change of variables formula:

$$\mathbb{E}[|X_1 \cdots X_n|] = \int_{\mathbb{R}^n} |x_1 \cdots x_n| \, \underbrace{\mu_{\mathbb{X}}(dx_1 \cdots dx_n)}_{\mu_{X_1} \otimes \cdots \otimes \mu_{X_n}}$$

$$= \int \cdots \left(\int |x_1| \cdots |x_n| \, \mu_{X_n}(dx_n) \right) \cdots \mu_{X_1}(dx_1)$$

$$= \int |x_1| \mu_{X_1}(dx_1) \cdots \int |x_n| \mu_{X_n}(dx_n) = \mathbb{E}[|X_1|] \cdots \mathbb{E}[|X_n|] < \infty.$$

Repeat, using Fubini. ///

Theorem: Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (S_i, \mathcal{B}_i)$ be rv's, $i \in [n]$.
Set $\mathbb{X} = (X_1, \dots, X_n)$. TFAE:

1. $X_1, \dots, X_n$ are independent.
2. $\mu_{\mathbb{X}} = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$
3. $\mathbb{E}[f_1(X_1) \cdots f_n(X_n)] = \mathbb{E}[f_1(X_1)] \cdots \mathbb{E}[f_n(X_n)]$ $(\star)$ $\quad \forall f_i \in B(S_i, \mathcal{B}_i)$.

Moreover, if each $(S_i, \mathcal{B}_i) = (\mathbb{R}^{d_i}, \mathcal{B}(\mathbb{R}^{d_i}))$, we also have the equivalent conditions

4. $(\star)$ holds $\forall f_i \in C_c(\mathbb{R}^{d_i})$
5. $(\star)$ holds $\forall f_i$ of the form $f_i = \mathbb{1}_{(-\infty, t_1] \times \cdots \times (-\infty, t_{d_i}]}$, $t_1, \dots, t_{d_i} \in \mathbb{R}$

Pf. We've already shown $1 \Leftrightarrow 2$. $2 \Rightarrow 3, 4, 5$ follow from c.o.v. + Fubini's theorem, much like the previous proposition. $4, 5 \Rightarrow 3$ follow from Dynkin's mult. syst. thm.

$3 \Rightarrow 1$: $f_i = \mathbb{1}_{B_i}$, $B_i \in \mathcal{B}_i$

$$\mathbb{E}[f_i(X_i)] = \mathbb{E}[\mathbb{1}_{B_i}(X_i)] = \int \mathbb{1}_{B_i} \, d\mu_{X_i} = \int_{B_i} d\mu_{X_i} = \mathbb{P}(X_i \in B_i)$$

$$\mathbb{E}[f_1(X_1) \cdots f_n(X_n)] = \int \underbrace{\mathbb{1}_{B_1}(x_1) \cdots \mathbb{1}_{B_n}(x_n)}_{\mathbb{1}_{B_1 \times \cdots \times B_n}} \mu_{\mathbb{X}}(dx_1 \cdots dx_n) = \mathbb{P}(\mathbb{X} \in B_1 \times \cdots \times B_n)$$

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Groupings and Functions

Lemma: If $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent σ-fields (over Ω), and $n = n_1 + n_2 + \cdots + n_k$, then

$$\mathcal{G}_1 = \sigma(\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{n_1}),\ \mathcal{G}_2 = \sigma(\mathcal{F}_{n_1+1} \cup \cdots \cup \mathcal{F}_{n_1+n_2}),\ \dots,\ \mathcal{G}_k = \sigma(\mathcal{F}_{n_1+\cdots+n_{k-1}+1} \cup \cdots \cup \mathcal{F}_n)$$

are independent σ-fields.

Pf. $\mathcal{G}_1 = \sigma(\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_m)$, $\mathcal{G}_2 = \sigma(\mathcal{F}_{m+1} \cup \cdots \cup \mathcal{F}_n)$

$$\left.\begin{aligned} \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_m &\supseteq \mathcal{C}_1 = \{A_1 \cap \cdots \cap A_m : A_i \in \mathcal{F}_i,\ 1 \le i \le m\} \\ \mathcal{F}_{m+1} \cup \cdots \cup \mathcal{F}_n &\supseteq \mathcal{C}_2 = \{A_{m+1} \cap \cdots \cap A_n : A_i \in \mathcal{F}_i,\ m+1 \le i \le n\}. \end{aligned}\right\} \pi\text{-systems. independent.}$$

$\Downarrow$ $\sigma(\mathcal{C}_1), \sigma(\mathcal{C}_2)$ indep.

$C_1 \in \mathcal{C}_1$ $\quad C_1 = A_1 \cap \cdots \cap A_m$

$C_2 \in \mathcal{C}_2$ $\quad C_2 = A_{m+1} \cap \cdots \cap A_n$

$\mathcal{F}_i \ni A_i = \Omega \cap \Omega \cap \cdots \cap A_i \cap \cdots \cap \Omega \in \mathcal{C}_1$

$$\begin{aligned} \therefore\ \mathbb{P}(C_1 \cap C_2) &= \mathbb{P}(A_1 \cap \cdots \cap A_m \cap A_{m+1} \cap \cdots \cap A_n) \\ &= \underbrace{\mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_m)}\underbrace{\mathbb{P}(A_{m+1})\cdots\mathbb{P}(A_n)} \\ &= \mathbb{P}(A_1 \cap \cdots \cap A_m)\,\mathbb{P}(A_{m+1} \cap \cdots \cap A_n) = \mathbb{P}(C_1)\mathbb{P}(C_2) \end{aligned}$$

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Cor: Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (S_i, \mathcal{B}_i)$ be independent, $i \in [n]$.

Let $n = n_1 + n_2 + \cdots + n_k$. Let

$f_j : (S_{n_1+\cdots+n_{j-1}+1} \times \cdots \times S_{n_1+\cdots+n_j}, \mathcal{B}_\circ \otimes \cdots \otimes \mathcal{B}_\circ) \to \mathbb{R}$ be measurable, $j \in [k]$.

Then $Y_j = f_j(X_{n_1+\cdots+n_{j-1}+1}, \ldots, X_{n_1+\cdots+n_j})$ are independent, $j \in [k]$.

Eg. If X_1, X_2, X_3, X_4, X_5 are independent, so are

$$X_1 + X_2, \quad X_3 X_4, \quad e^{X_5}$$

Pf. $X_1, \ldots, X_m, \quad X_{m+1}, \ldots, X_n$

$Y_1 = f_1(X_1, \ldots, X_m) \qquad Y_2 = f_2(X_{m+1}, \ldots, X_n)$

$\sigma(Y_1) \subseteq \sigma(X_1, \ldots, X_m) \qquad \sigma(Y_2) \subseteq \sigma(X_{m+1}, \ldots, X_n)$

(Doob-Dynkin)

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Uncorrelated vs. Independent

E.g. $(X,Y) = (X, XZ)$, X, Z independent, $X \in L^2$, $|Z| \le 1$ with $\mathbb{E}[Z] = 0$.

Then $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$$= \mathbb{E}[X^2 Z] - \mathbb{E}[X]\mathbb{E}[XZ]$$

$$= \mathbb{E}[X^2]\cancel{\mathbb{E}[Z]} - \mathbb{E}[X]\mathbb{E}[X]\cancel{\mathbb{E}[Z]} = 0.$$

if $X^{n+m} \in L^1$

$$\mathbb{E}[X^n Y^m] = \mathbb{E}[X^n X^m Z^m] = \mathbb{E}[X^{n+m} Z^m]$$

$$= \mathbb{E}[X^{n+m}]\mathbb{E}[Z^m].$$

$$\mathbb{E}[X^n]\mathbb{E}[Y^m] = \mathbb{E}[X^n]\mathbb{E}[X^m]\mathbb{E}[Z^m]$$ $X^n, X^m \in L^1$.

$X \overset{d}{=} \mathcal{N}(0,1)$ $\qquad Z \overset{d}{=} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$

$$\mathbb{E}[X^{2+2}] = (4-1)!! = 3 \cdot 1 = 3.$$

$$\mathbb{E}[X^2]\mathbb{E}[X^2] = 1 \cdot 1 = 1$$

Method of Moments

Proposition: Let $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be bounded rv's $i \in [n]$

$\exists M < \infty$ $|X_1|, \dots, |X_n| \leq M$ a.s.

Then $X_1, \dots, X_n$ are independent iff $|X_j^k| \leq M^k$ a.s.

$$\mathbb{E}[X_1^{k_1} \cdots X_n^{k_n}] = \mathbb{E}[X_1^{k_1}] \cdots \mathbb{E}[X_n^{k_n}], \quad \forall k_1, \dots, k_n \in \mathbb{N}.$$

Pf. $(\Rightarrow)$ $X_1^{k_1}, \dots, X_n^{k_n}$ are independent, L^1 ✓

$(\Leftarrow)$ [HW].