Conditioning

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space,
and $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$.

$$\mathbb{P}(\,\cdot\,|B) : \mathcal{F} \to [0,1], \quad \mathbb{P}(A|B) :=$$

is another probability measure on $(\Omega, \mathcal{F})$.
It is ==conditional probability==: $\mathbb{P}(A|B)$ is the "new" probability of event A, in the event that B has occurred.

Eg. Toss a fair coin twice.

$$\Omega = \{HH, HT, TH, TT\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(A) = \frac{\#A}{4}.$$

$$\mathbb{P}(\text{Second toss is } H \mid \text{First toss is } H)$$

Independence

Events $A, B \in \mathcal{F}$ are (statistically) independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

More generally, if $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{F}$ are two collections of events, we say they are independent if $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ $\forall A_1 \in \mathcal{C}_1, A_2 \in \mathcal{C}_2$.

It will be customary to apply this with $\mathcal{C}_j$ σ-fields; if so, we can recover the original definition by applying it to $\mathcal{C}_j = \sigma\{A_j\}$

Observation: If A, B are independent, so are $\sigma(A), \sigma(B)$.

Independence of Many Collections of Events

What should it mean for $A, B, C \in \mathcal{F}$ to be independent?

Maybe just pairwise independence?

Eg. Two fair coin tosses again. $A = \{HH, HT\}$ "first toss is H"

$B = \{HH, TH\}$ "second toss is H"

$C = \{HT, TH\}$ "the two tosses don't agree"

$\mathbb{P}(A \cap B) =$

$\mathbb{P}(A \cap C) =$

$\mathbb{P}(B \cap C) =$

$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$

$\therefore \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(C) = \mathbb{P}(B)\mathbb{P}(C) = \frac{1}{4}$

But these should not be "independent", since A & $B \Rightarrow \neg C$!

$\mathbb{P}(A \cap B \cap C) =$

Maybe we just want $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$?

E.g. Take <u>any</u> events A, B, and set $C = \phi$.

<u>Def</u>: $\mathcal{C}_1, \dots, \mathcal{C}_n \subseteq \mathcal{F}$ are **independent** if: $\forall I \subseteq [n]$

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i), \quad \forall A_i \in \mathcal{C}_i, \quad i \in I.$$

<u>Observation</u>: If $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, so are $\mathcal{C}_1 \cup \Omega, \dots, \mathcal{C}_n \cup \Omega$.
This makes the notation <u>so</u> much easier.

<u>Lemma</u>: If $\mathcal{C}_1, \dots, \mathcal{C}_n \subseteq \mathcal{F}$ and $\Omega \in \mathcal{C}_j$ for all $j \in [n]$ then

Independence and σ-Fields

We saw that events A, B being independent

$\Rightarrow$ $\sigma(A), \sigma(B)$ are independent.

This does not apply to collections. [HW]

But it does if the collections are closed under finite intersections.

Def: A collection $\mathcal{C} \subseteq \mathcal{F}$ is a π-system if it is closed under finite intersections:

$$A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$$

Theorem: [15.2] If $\mathcal{C}_1, \ldots, \mathcal{C}_n \subseteq \mathcal{F}$ are independent π-systems, then $\sigma(\mathcal{C}_1), \ldots, \sigma(\mathcal{C}_n)$ are independent.

Lemma: If $\mathcal{C} \subseteq \mathcal{F}$ is a π-system, and μ, ν are probability measures on $\mathcal{F}$ s.t. $\mu = \nu$ on $\mathcal{C}$, then $\mu = \nu$ on $\sigma(\mathcal{C})$.

Pf.

Theorem: [15.2] If $\mathcal{C}_1, \dots, \mathcal{C}_n \subseteq \mathcal{F}$ are independent π-systems then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Pf.

Def: Let $\{\mathcal{C}_t\}_{t\in T}$ be any collection of subsets of $\mathcal{F}$.
Call them *independent* iff, for all finite subsets $J \subset T$, $\{\mathcal{C}_j\}_{j\in J}$ is independent.

Lemma: Let $\{A_n\}_{n=1}^{\infty}$ be an infinite sequence of independent events. Then

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathbb{P}(A_n) \qquad [HW]$$

Borel-Cantelli Lemma (II)

Let $\{A_n\}_{n=1}^{\infty}$ be independent events. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\{A_n \text{ i.o.}\}) = 1$.

E.g. $X_n \overset{d}{=} \text{Bernoulli}(p_n)$ where $\sum_{n=1}^{\infty} p_n = \infty$ (e.g. $p_n = \frac{1}{n}$).
If the events $\{X_n = 1\}$ are all independent (e.g. tossing a sequence of biased independent coins with $\mathbb{P}(\text{Heads}) = p_n$), then $\mathbb{P}(X_n = 1 \text{ for } \infty\text{-many } n) = 1$.

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Pf. $\{A_n \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$