Conditioning

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$.

$$\mathbb{P}(\cdot \mid B) : \mathcal{F} \to [0,1], \quad \mathbb{P}(A \mid B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is another probability measure on $(\Omega, \mathcal{F})$.

It is conditional probability: $\mathbb{P}(A \mid B)$ is the "new" probability of event A, in the event that B has occurred.

E.g. Toss a fair coin twice.

$$\Omega = \{HH, HT, TH, TT\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(A) = \frac{\#A}{4}.$$

$$\mathbb{P}(\underbrace{\text{Second toss is H}}_{A} \mid \underbrace{\text{First toss is H}}_{B}) = \frac{\mathbb{P}(\{HH, TH\} \cap \{HH, HT\})}{\mathbb{P}(\{HH, HT\})} = \frac{\mathbb{P}\{HH\}}{\mathbb{P}\{HH, HT\}} = \frac{1/4}{1/2} = \frac{1}{2}.$$

$$\mathbb{P}(\underbrace{\text{Second toss in H}}_{\{HH, TH\}}) = \frac{1}{2}.$$

$$\left.\begin{array}{c}\mathbb{P}(A \mid B) = \mathbb{P}(A) \\ \mathbb{P}(A \cap B) / \mathbb{P}(B)\end{array}\right\} \quad \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Independence

Events $A, B \in \mathcal{F}$ are (statistically) independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

$$\big(\ \mathbb{P}(B) > 0,\quad \mathbb{P}(A|B) = \mathbb{P}(A)\ ;\quad \mathbb{P}(A) > 0,\quad \mathbb{P}(B|A) = \mathbb{P}(B).\ \big)$$

More generally, if $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{F}$ are two collections of events, we say they are independent if $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ $\forall A_1 \in \mathcal{C}_1, A_2 \in \mathcal{C}_2$.

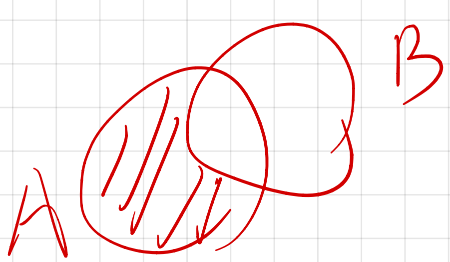
It will be customary to apply this with $\mathcal{C}_j$ σ-fields; if so, we can recover the original definition by applying it to $\mathcal{C}_j = \sigma\{A_j\} = \{\emptyset, A_j, A_j^c, \Omega\}$

Observation: If A, B are independent, so are $\sigma(A), \sigma(B)$.

$A, \emptyset$ are indep.

$$\begin{cases} \mathbb{P}(A \cap \Omega) = \mathbb{P}(A) = \mathbb{P}(A)\cdot 1 = \mathbb{P}(A)\,\mathbb{P}(\Omega) \quad \checkmark \\ \mathbb{P}(A \cap B^c) = \mathbb{P}(A \setminus A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) \end{cases}$$

$$= \mathbb{P}(A) - \mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A)\,\mathbb{P}(B^c).$$



Independence of Many Collections of Events

What should it mean for $A, B, C \in \mathcal{F}$ to be independent?

Maybe just pairwise independence?

Eg. Two fair coin tosses again. $A = \{HH, HT\}$ "first toss is H"

$B = \{HH, TH\}$ "second toss is H"

$C = \{HT, TH\}$ "the two tosses don't agree"

$\mathbb{P}(A \cap B) = \mathbb{P}\{HH\} = \frac{1}{4}$ ✓

$\mathbb{P}(A \cap C) = \mathbb{P}\{HT\} = \frac{1}{4}$ ✓

$\mathbb{P}(B \cap C) = \mathbb{P}\{TH\} = \frac{1}{4}$ ✓

$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$

$\therefore\ \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(C) = \mathbb{P}(B)\mathbb{P}(C) = \frac{1}{4}$

But these should not be "independent", since A & $B \Rightarrow \neg C$!

I.e. $A \cap B \subseteq C^c$ i.e. $A \cap B \cap C = \emptyset$

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\emptyset) = 0 \neq \frac{1}{8} = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

Maybe we just want $\mathbb{P}(\underbrace{A\cap B}_{\emptyset}\cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(\cancel{C})$?

$\searrow 0 \qquad\qquad \downarrow 0$

E.g. Take <u>any</u> events A, B, and set $C = \emptyset$.

Def: $\mathcal{C}_1, \dots, \mathcal{C}_n \subseteq \mathcal{F}$ are **independent** if: $\forall I \subseteq [n] = \{1, \dots, n\}$

$$\mathbb{P}\Big(\bigcap_{i\in I} A_i\Big) = \prod_{i\in I} \mathbb{P}(A_i), \quad \forall A_i \in \mathcal{C}_i, \quad i \in I.$$

$n=3$: $\mathbb{P}(A_i) = \mathbb{P}(A_i)$ ✓ $\quad \underset{i\neq j}{\mathbb{P}(A_i \cap A_j)} = \mathbb{P}(A_i)\mathbb{P}(A_j)$, $\mathbb{P}(A_1\cap A_2\cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$

Observation: If $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, so are $\mathcal{C}_1 \cup \Omega, \dots, \mathcal{C}_n \cup \Omega$.

This makes the notation <u>so</u> much easier.

E.g. $n=5$ $\qquad A_1 \cap A_3 \cap A_4 = A_1 \cap \Omega \cap A_3 \cap A_4 \cap \Omega$

$\mathbb{P}(A_1 \cap A_3 \cap A_4) = \mathbb{P}(A_1)\mathbb{P}(A_3)\mathbb{P}(A_4) = \mathbb{P}(A_1)\mathbb{P}(\Omega)\mathbb{P}(A_3)\mathbb{P}(A_4)\mathbb{P}(\Omega)$

Lemma: If $\underbrace{\mathcal{C}_1, \dots, \mathcal{C}_n}_{\text{independent}} \subseteq \mathcal{F}$ and $\Omega \in \mathcal{C}_j$ for all $j \in [n]$ then iff $\mathbb{P}(A_1 \cap \dots \cap A_n) = \prod_{i=1}^{n} \mathbb{P}(A_i) \quad \forall A_i \in \mathcal{C}_i \ni \Omega$

Independence and σ-Fields

We saw that events A, B being independent $\Rightarrow$ $\sigma(A), \sigma(B)$ are independent.

This does not apply to collections. [HW]
But it does if the collections are closed under finite intersections.

Def: A collection $\mathcal{C} \subseteq \mathcal{F}$ is a π-system if it is closed under finite intersections:

$$A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$$

Theorem: [15.2] If $\mathcal{C}_1, \dots, \mathcal{C}_n \subseteq \mathcal{F}$ are independent π-systems, then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Lemma: If $\mathcal{C} \subseteq \mathcal{F}$ is a π-system, and μ, ν are probability measures on $\mathcal{F}$ s.t. $\mu = \nu$ on $\mathcal{C}$, then $\mu = \nu$ on $\sigma(\mathcal{C})$.

Pf. $\mathbb{M} = \{ \mathbb{1}_B : B \in \mathcal{C} \} \subseteq \mathbb{B}(\mathcal{F})$

$\mathbb{1}_A \cdot \mathbb{1}_B = \mathbb{1}_{A \cap B} \in \mathbb{M}$. i.e. $\mathbb{M}$ is a mult. system.

$\mathbb{H} = \{ f \in \mathbb{B}(\mathcal{F}) : \int f \, d\mu = \int f \, d\nu \}$

- $1 \in \mathbb{H}$ ✓
- $\mathbb{H}$ is a subspace ✓
- $\mathbb{H}$ is closed under bdd convergence ✓
- $\mathbb{M} \subseteq \underline{\mathbb{H}}$: $\int \mathbb{1}_B \, d\mu = \mu(B) = \nu(B) = \int \mathbb{1}_B \, d\nu$ ✓

$f^{-1}(E) = \mathbb{1}_B^{-1}\{1\} = B$

$\therefore$ By Dynkin: $\mathbb{B}(\sigma(\mathbb{M})) \subseteq \mathbb{H}$.

$\mathbb{1}_A \in \mathbb{B}(\sigma(\mathbb{M}))$, $A \in \sigma(\mathbb{M}) = \sigma\{ f^{-1}(E) : E \in \mathcal{B}(\mathbb{R}),\ f \in \mathbb{M} \} \supseteq \sigma(\mathcal{C})$.

$E = \{1\}$, $f = \mathbb{1}_B$, $B \in \mathcal{C}$

Theorem: [15.2] If $\mathcal{C}_1, \dots, \mathcal{C}_n \subseteq \mathcal{F}$ are independent π-systems then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Pf. $n=2$; general by induction.

Fix $B \in \mathcal{C}_2$: $\mathbb{P}(B) = 0$, $\mathbb{P}(A \cap B) \le \mathbb{P}(B) = 0 = \mathbb{P}(A)\mathbb{P}(B)$ $\forall A \in \mathcal{F}$, $A \in \sigma(\mathcal{C}_1)$

If $\mathbb{P}(B) > 0$, $\mathbb{P}(\cdot \mid B)$ is a prob. meas. on $\mathcal{F}$

$\mathbb{P}$ " on $\underline{\mathcal{C}}_1$ i.e. $\mathbb{P}(A) = \mathbb{P}(A \mid B)$ $\forall A \in \mathcal{C}_1$

$\therefore$ by Lemma, $\Rightarrow$ $\mathbb{P}(\ \mid B) = \mathbb{P}$ agree on $\sigma(\mathcal{C}_1)$.

i.e. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ $\forall A \in \sigma(\mathcal{C}_1)$, $B \in \mathcal{C}_2$.

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Def: Let $\{\mathcal{C}_t\}_{t\in T}$ be any collection of subsets of $\mathcal{F}$. Call them *independent* iff, for all finite subsets $J \subset T$, $\{\mathcal{C}_j\}_{j\in J}$ is independent.

Lemma: Let $\{A_n\}_{n=1}^{\infty}$ be an infinite sequence of independent events. Then

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathbb{P}(A_n) := \lim_{M\to\infty} \prod_{n=1}^{M} \mathbb{P}(A_n) \quad [HW]$$

Borel-Cantelli Lemma (II)

Let $\{A_n\}_{n=1}^{\infty}$ be independent events. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\{A_n \text{ i.o.}\}) = 1$.

E.g. $X_n \overset{d}{=} \text{Bernoulli}(p_n)$ where $\sum_{n=1}^{\infty} p_n = \infty$ (e.g. $p_n = \frac{1}{n}$).
If the events $\{X_n = 1\}$ are all independent (e.g. tossing a sequence of biased independent coins with $\mathbb{P}(\text{Heads}) = p_n$), then $\mathbb{P}(X_n = 1 \text{ for } \infty\text{-many } n) = 1$.

Borel - Cantelli Lemma (II)

Let $\{A_n\}_{n=1}^{\infty}$ be independent events. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\{A_n \text{ i.o.}\}) = 1$.

Pf. $\{A_n \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \underbrace{\bigcup_{n \geq k} A_n}_{U_k \downarrow}$

$$\therefore \mathbb{P}(A_n \text{ i.o.}) = \lim_{k\to\infty} \mathbb{P}\Big(\bigcup_{n\geq k} A_n\Big) = \lim_{k\to\infty} \Big(1 - \mathbb{P}\Big(\bigcap_{n\geq k} A_n^c\Big)\Big)$$

$$= 1 - \lim_{k\to\infty} \mathbb{P}\Big(\bigcap_{n\geq k} A_n^c\Big) = 1 - 0 = 1$$

$$\mathbb{P}\Big(\bigcap_{n\geq k}^{\infty} A_n^c\Big) = \prod_{n=k}^{\infty} \mathbb{P}(A_n^c) = \prod_{n=k}^{\infty}(1 - \underbrace{\mathbb{P}(A_n)}_{p_n \in [0,1]})$$

$0 \leq 1 - p_n \leq e^{-p_n}$

$\begin{cases} f(p) = e^{-p} + p - 1, \quad f(0) = 0 \\ f'(p) = -e^{-p} + 1 \geq 0 \quad \text{on } [0,1] \end{cases}$

$$\leq \prod_{n\geq k} e^{-p_n}$$

$$= \lim_{M\to\infty} \prod_{n=k}^{M} e^{-p_n}$$

$$= \lim_{M\to\infty} e^{-\sum_{n\geq k}^{M} p_n} = 0 \quad \forall k.$$

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