We have now constructed **product measure**:

$(\Omega_j, \mathcal{F}_j, \mu_j)$ (σ-finite) $\rightsquigarrow$ $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$

$$\mu_1 \otimes \mu_2 (E) = \int_{\Omega_1} \left(\int_{\Omega_2} \mathbb{1}_E(\omega_1, \omega_2) \, \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$$

So, how do we integrate a function against $\mu_1 \otimes \mu_2$?

Theorem: (Tonelli) Let $f \geq 0$ be $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$-measurable.

Then
$$\int_{\Omega_1 \times \Omega_2} f \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) \, \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$$

Pf.

Eg. (The right integration constant for $\mathcal{N}(0,1)$)

$$f(x) = e^{-x^2/2}$$

$$I := \int_{\mathbb{R}} f(x)\,\lambda(dx).$$

$$I^2 = \int_{\mathbb{R}} f(x)\,\lambda(dx) \cdot \int_{\mathbb{R}} f(y)\,\lambda(dy) =$$

Theorem: (Fubini) Let $f \in L^0(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$.

TFAE:

1. $f \in L^1(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$

2. $\int_{\Omega_1} \left(\int_{\Omega_2} |f(\omega_1, \omega_2)| \, \mu_2(d\omega_2) \right) \mu_1(d\omega_1) < \infty$

3. $\int_{\Omega_2} \left(\int_{\Omega_1} |f(\omega_1, \omega_2)| \, \mu_1(d\omega_1) \right) \mu_2(d\omega_2) < \infty$

In this case, $\omega_1 \mapsto f(\omega_1, \omega_2) \in L^1(\Omega_1, \mathcal{F}_1, \mu_1)$ for $[\mu_2]$-a.e. ω_2,

$\omega_2 \mapsto f(\omega_1, \omega_2) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$ for $[\mu_1]$-a.e. ω_1,

$$\omega_2 \mapsto \int f(\omega_1, \omega_2) \, \mu_1(d\omega_1) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2),$$

$$\omega_1 \mapsto \int f(\omega_1, \omega_2) \, \mu_2(d\omega_2) \in L^1(\Omega_1, \mathcal{F}_1, \mu_1),$$

and
$$\int_{\Omega_1 \times \Omega_2} f \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) \, \mu_2(d\omega_2) \right) \mu_1(d\omega_1).$$

Pf. Let $E_1 = \{ \omega_1 \in \Omega_1 : \int_{\Omega_2} |f(\omega_1, \omega_2)| \mu_2(d\omega_2) = \infty \}$

Notation: $⨍ g \, d\nu = \begin{cases} \int g \, d\nu & \text{if } g \in L^1(\nu) \\ 0 & \text{otherwise} \end{cases}$

$$\int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) = \int_{\Omega_2} \mathbb{1}_{E^c}(\omega_1) f(\omega_1, \omega_2)\, \mu_2(d\omega_2)$$

$$\int_{\Omega_1} \left| \int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right| \mu_1(d\omega_1)$$

Finally: $\displaystyle\int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1,\omega_2)\,\mu_2(d\omega_2) \right) \mu_1(d\omega_1)$

$$= \int_{\Omega_1} \mu_1(d\omega_1) \left(\int_{\Omega_2} \mu_2(d\omega_2)\, \mathbb{1}_{E^c}(\omega_1)\, f_+(\omega_1,\omega_2) - \int_{\Omega_2} \mu_2(d\omega_2)\, \mathbb{1}_{E^c}(\omega_1)\, f_-(\omega_1,\omega_2) \right)$$

$$= \int_{\Omega_2} \mu_2(d\omega_2) \int_{\Omega_1} \mu_1(d\omega_1)\, \mathbb{1}_{E^c}(\omega_1)\, f_+(\omega_1,\omega_2) - \int_{\Omega_2} \mu_2(d\omega_2) \int_{\Omega_1} \mu_1(d\omega_1)\, \mathbb{1}_{E^c}(\omega_1)\, f_+(\omega_1,\omega_2)$$