We have now constructed product measure:

$(\Omega_j, \mathcal{F}_j, \mu_j)$ ($\sigma$-finite) $\rightsquigarrow$ $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$

$$\mu_1 \otimes \mu_2 (E) = \int_{\Omega_1} \left( \int_{\Omega_2} 1\!\!1_E(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$$

So, how do we integrate a function against $\mu_1 \otimes \mu_2$?

**Theorem:** (Tonelli) Let $f \geq 0$ be $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$-measurable.

Then

$$\int_{\Omega_1 \times \Omega_2} f \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left( \int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1) \quad (\star)$$

Pf. $\star$ holds for $f = 1\!\!1_E$. $\therefore$ By linearity, holds for $\mathcal{F}_1 \otimes \mathcal{F}_2$-simple fns.

$\varphi_n \uparrow f$ $\Rightarrow$ $\star$ by 3 apps of MCT.

$\mathcal{F}_1 \otimes \mathcal{F}_2$-simple fns.

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Eg. (The right integration constant for $\mathcal{N}(0,1)$)

$$f(x) = e^{-x^2/2} \geq 0 \qquad \text{(continuous)}$$

$$I := \int_{\mathbb{R}} f(x)\, \lambda(dx) \; = \sqrt{2\pi}$$

$$I^2 = \int_{\mathbb{R}} f(x)\, \lambda(dx) \cdot \int_{\mathbb{R}} f(y)\, \lambda(dy) = \int_{\mathbb{R}\times\mathbb{R}} f\otimes f \; d(\lambda\otimes\lambda) \overset{?}{=} 2\pi$$

$$f\otimes f\,(x,y) = e^{-x^2/2}\, e^{-y^2/2} = e^{-\frac{x^2}{2} - \frac{y^2}{2}} = e^{-\frac{1}{2}(x^2+y^2)}$$

$$\bar{D}_R = \{(x,y) : x^2+y^2 \leq R\}, \text{ then } \bar{D}_R \uparrow \mathbb{R}^2 \text{ as } R\uparrow\infty$$

$$\therefore \underbrace{\lim_{R\uparrow\infty} \int_{\bar{D}_R} f\otimes f \; d(\lambda\otimes\lambda)}_{\|} = \int_{\mathbb{R}^2} f\otimes f \; d(\lambda\otimes\lambda) \qquad \text{(by MCT)}$$

$$\iint_{\bar{D}_R} e^{-\frac{1}{2}(x^2+y^2)}\, dx\,dy = \int_0^{2\pi} d\theta \int_0^R r\,dr\; e^{-r^2/2} = 2\pi\left(1-e^{-R^2/2}\right) \xrightarrow[R\to\infty]{} 2\pi.$$

Theorem: (Fubini) Let $f \in L^0(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$.

TFAE:

1. $f \in L^1(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$

Tonelli: $\int_{\Omega_1 \times \Omega_2} |f| \, d(\mu_1 \otimes \mu_2) =$

2. $\displaystyle\int_{\Omega_1} \left( \int_{\Omega_2} |f(\omega_1, \omega_2)| \, \mu_2(d\omega_2) \right) \mu_1(d\omega_1) < \infty$

3. $\displaystyle\int_{\Omega_2} \left( \int_{\Omega_1} |f(\omega_1, \omega_2)| \, \mu_1(d\omega_1) \right) \mu_2(d\omega_2) < \infty$

In this case, $\omega_1 \mapsto f(\omega_1, \omega_2) \in L^1(\Omega_1, \mathcal{F}_1, \mu_1)$ for $[\mu_2]$-a.e. $\omega_2$,

$\omega_2 \mapsto f(\omega_1, \omega_2) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$ for $[\mu_1]$-a.e. $\omega_1$,

modify $f$ on a nullset

$$\omega_2 \mapsto \cancel{\int} f(\omega_1, \omega_2) \, \mu_1(d\omega_1) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2),$$

$$\omega_1 \mapsto \cancel{\int} f(\omega_1, \omega_2) \, \mu_2(d\omega_2) \in L^1(\Omega_1, \mathcal{F}_1, \mu_1),$$

and

$$\int_{\Omega_1 \times \Omega_2} f \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left( \int_{\Omega_2} f(\omega_1, \omega_2) \, \mu_2(d\omega_2) \right) \mu_1(d\omega_1).$$

Pf. Let $E_1 = \{ \omega_1 \in \Omega_1 : \underbrace{\int_{\Omega_2} |f(\omega_1, \omega_2)| \mu_2(d\omega_2)}_{} = \infty \} = I^{-1}(\{\infty\})$

By Tonelli, $I(\omega_1)$ is $\mathcal{F}_1 / \mathcal{B}(\overline{\mathbb{R}})$-meas. $\therefore E_1 \in \mathcal{F}_1$

Since $f \in L^1(\mu_1 \otimes \mu_2)$, $\int_{\Omega_1} \underset{\substack{\vee\!\!\vee \\ 0}}{I(\omega_1)} \mu_1(d\omega_1) = \int_{\Omega_1 \times \Omega_2} |f| \, d(\mu_1 \otimes \mu_2) < \infty$.

$\therefore I < \infty$ $\mu_1$-a.s. $\Rightarrow \mu_1(E_1) = 0$.

Notation: $\oint g \, d\nu = \begin{cases} \int g \, d\nu & \text{if } g \in L^1(\nu) \\ 0 & \text{otherwise} \end{cases}$

$$\int_{\Omega_2} f(\omega_1,\omega_2)\,\mu_2(d\omega_2) = \int_{\Omega_2} \mathbb{1}_{E_1^c}(\omega_1) f(\omega_1,\omega_2)\,\mu_2(d\omega_2)$$

$$\mathbb{1}_{E_1^c}(\omega_1) = \mathbb{1}_{E^c} \otimes 1\,(\omega_1,\omega_2)$$

$$= \int_{\Omega_2} \mathbb{1}_{E_1^c}(\omega_1)\left[f_+(\omega_1,\omega_2) - f_-(\omega_1,\omega_2)\right]\mu_2(d\omega_2)$$

$$= \int_{\Omega_2} \mathbb{1}_{E_1^c}(\omega_1) f_+(\omega_1,\omega_2)\,\mu_2(d\omega_2) - \int_{\Omega_2} \mathbb{1}_{E_1^c}(\omega_1) f_-(\omega_1,\omega_2)\,\mu_2(d\omega_2)$$

$\therefore$ by Tonelli, $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1,\omega_2)\,\mu_2(d\omega_2)$

is $\mathcal{F}_1/\mathcal{B}(\mathbb{R})$-meas.

Moreover:

$$\int_{\Omega_1} \left| \int_{\Omega_2} f(\omega_1,\omega_2)\,\mu_2(d\omega_2)\right| \mu_1(d\omega_1) \leq \int_{\Omega_1}\left(\int_{\Omega_2} \mathbb{1}_{E_1^c}(\omega_1)\,|f(\omega_1,\omega_2)|\,\mu_2(d\omega_2)\right)\mu_1(d\omega_1)$$

$$= \int_{E_1^c\times\Omega_2} |f|\, d(\mu_1\otimes\mu_2) < \infty.$$

Finally: $\displaystyle\int_{\Omega_1}\left(\int_{\Omega_2} f(\omega_1,\omega_2)\,\mu_2(d\omega_2)\right)\mu_1(d\omega_1)$

$$= \int_{\Omega_1}\mu_1(d\omega_1)\left(\int_{\Omega_2}\mu_2(d\omega_2)\,\mathbb{1}_{E^c}(\omega_1)\, f_+(\omega_1,\omega_2) - \int_{\Omega_2}\mu_2(d\omega_2)\,\mathbb{1}_{E^c}(\omega_1)\, f_-(\omega_1,\omega_2)\right)$$

$$= \int_{\Omega_2}\mu_2(d\omega_2)\int_{\Omega_1}\mu_1(d\omega_1)\,\mathbb{1}_{E^c}(\omega_1)\, f_+(\omega_1,\omega_2) - \int_{\Omega_2}\mu_2(d\omega_2)\int_{\Omega_1}\mu_1(d\omega_1)\,\mathbb{1}_{E^c}(\omega_1)\, f_+(\omega_1,\omega_2)$$

$$= \int_{\Omega_1\cancel{E^c}\times\Omega_2} f_+ \, d(\mu_1\otimes\mu_2) - \int_{\Omega_2\cancel{E_1^c}\times\Omega_2} f_- \, d(\mu_1\otimes\mu_2) \quad \text{(by Tonelli)}$$

$$= \int_{\Omega_1\otimes\Omega_2} f \, d(\mu_1\otimes\mu_2)$$

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