Product Measure

Eg. $\mathcal{B}(\mathbb{R}^d) = \sigma\{(a_1,b_1]\times(a_2,b_2]\times\cdots\times(a_d,b_d] : -\infty\le a_j\le b_j\le\infty\}$

Def Given two measurable spaces $(\Omega_1,\mathcal{F}_1),(\Omega_2,\mathcal{F}_2)$,

$$\mathcal{F}_1\otimes\mathcal{F}_2 :=$$

By induction, larger products are

$$\bigotimes_{j=1}^{d}\mathcal{F}_j =$$

Thus $\mathcal{B}(\mathbb{R}^d) = \bigotimes_{j=1}^{d}\mathcal{B}(\mathbb{R})$.

Lemma: Let $\pi_k : \prod_{j=1}^{d}\Omega_j \to \Omega_k$ be the standard projection: $\pi_k(\omega_1,\omega_2,\dots,\omega_d)=\omega_k$.

Then $\bigotimes_{j=1}^{d}\mathcal{F}_j = \sigma\{\pi_k : 1\le k\le d\}$.

Lemma: (Product Measurability)

Let $(\Omega_j, \mathcal{F}_j)_{j\in J}$ and $(\Upsilon, \mathcal{B})$ be measurable spaces.

Then $f : \Upsilon \to \prod_{j\in J} \Omega_j$ is $\mathcal{B}/\bigotimes_{j\in J} \mathcal{F}_j$ - measurable

iff $\pi_k \circ f : \Upsilon \to \Omega_k$ is $\mathcal{B}/\mathcal{F}_k$ - measurable $\forall k \in J$.

Pf. We use the fact that $\bigotimes_{j\in J} \mathcal{F}_j = \sigma\{\pi_j : j \in J\}$.

$(\Rightarrow)$

$(\Leftarrow)$

Eg. Let $f_j : \Omega_j \to \mathbb{R}$ be measurable functions. Then

$f_1 \otimes f_2 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is defined by

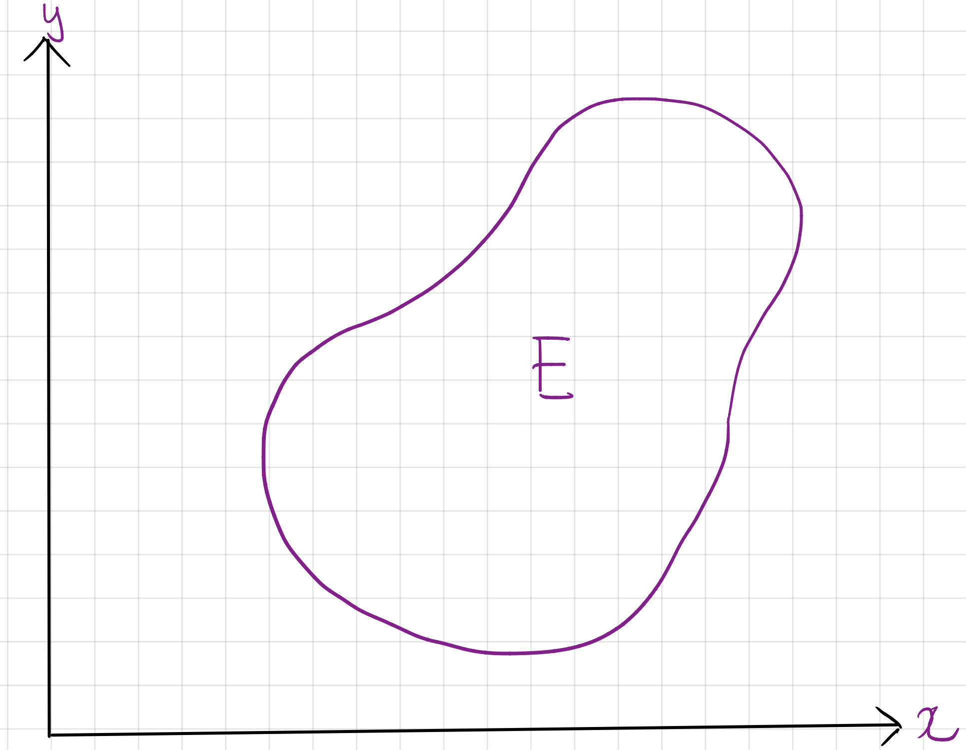
$(f_1 \otimes f_2)(\omega_1, \omega_2) :=$

It is $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$ - measurable.

We will use such "tensor product" functions frequently in our construction of **product measure**.

If $(\Omega_1, \mathcal{F}_1, \mu_1)$, $(\Omega_2, \mathcal{F}_2, \mu_2)$ are measure spaces, we want to construct a measure $\mu_1 \otimes \mu_2$ on $\mathcal{F}_1 \otimes \mathcal{F}_2$ satisfying

$$(\mu_1 \otimes \mu_2)(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2) \quad \text{for } B_j \in \mathcal{F}_j.$$



To really understand product measure,
we need to figure out how to do

Theorem: [13.3] Let $(\Omega_j, \mathcal{F}_j, \mu_j)$ $j=1,2$ be σ-finite measure spaces. Let $f: \Omega_1 \times \Omega_2 \to [0,\infty)$ be a non-negative $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$-measurable function. Then:

1. (a) $\omega_1 \mapsto f(\omega_1, \omega_2)$ is $\mathcal{F}_1 / \mathcal{B}(\mathbb{R})$-measurable $\forall \omega_2 \in \Omega_2$
   (b) $\omega_2 \mapsto f(\omega_1, \omega_2)$ is $\mathcal{F}_2 / \mathcal{B}(\mathbb{R})$-measurable $\forall \omega_1 \in \Omega_1$

2. (a) $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2)$ is $\mathcal{F}_1 / \mathcal{B}(\overline{\mathbb{R}})$-measurable.
   (b) $\omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2)\, \mu_1(d\omega_1)$ is $\mathcal{F}_2 / \mathcal{B}(\overline{\mathbb{R}})$-measurable.

3. $$\int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1) = \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2)\, \mu_1(d\omega_1) \right) \mu_2(d\omega_2)$$

We'll prove this for μ_1, μ_2 finite measures; the extension to σ-finite is standard

Pf. Step 1. Verify that 1-3 hold for $f = f_1 \otimes f_2$, $f_j \in B(\Omega_j, \mathcal{F}_j)$

1. (a) $\omega_1 \mapsto f(\omega_1, \omega_2)$

   (b)

2. (a) $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2)$

   (b)

3. $\int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$

Step 2. Let $\mathbb{H} = \{ f \in \mathbb{B}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) : 1\text{-}3 \text{ hold} \}$

Let $\mathbb{M} = \mathbb{B}(\Omega_1, \mathcal{F}_1) \otimes \mathbb{B}(\Omega_2, \mathcal{F}_2)$

- $\mathbb{M}$ is a multiplicative system:

- $\sigma(\mathbb{M}) = \mathcal{F}_1 \otimes \mathcal{F}_2$

- $\mathbb{H} \supseteq \mathbb{M} \cup \{1\}$
- $\mathbb{H}$ is closed under bounded convergence.

Thus, by Dynkin's Multiplicative Systems Theorem,

<u>Step 3</u>. Let $f \geq 0$ be $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$ - measurable.

For $n \in \mathbb{N}$, set $f_n = \min\{f, n\}$.

Cor: There is a unique measure $\mu_1 \otimes \mu_2$ on $\mathcal{F}_1 \otimes \mathcal{F}_2$ s.t. $\mu_1 \otimes \mu_2 (E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$ for all $E_j \in \mathcal{F}_j$, and it is given by

$$(\mu_1 \otimes \mu_2)(E) = \int_{\Omega_1} \left(\int_{\Omega_2} \mathbb{1}_E(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$$
$$= \int_{\Omega_2} \left(\int_{\Omega_1} \mathbb{1}_E(\omega_1, \omega_2)\, \mu_1(d\omega_1) \right) \mu_2(d\omega_2) \qquad \forall E \in \mathcal{F}_1 \otimes \mathcal{F}_2.$$

for $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

Pf.