Product Measure

Eg. $\mathcal{B}(\mathbb{R}^d) = \sigma\{(a_1,b_1]\times(a_2,b_2]\times\cdots\times(a_d,b_d] : -\infty\le a_j\le b_j\le\infty\}$

includes $B_1\times B_2\times\cdots\times B_d$, $B_j\in\mathcal{B}(\mathbb{R})$

$= \sigma(\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R})\times\cdots\times\mathcal{B}(\mathbb{R}))$.

Def Given two measurable spaces $(\Omega_1,\mathcal{F}_1)$, $(\Omega_2,\mathcal{F}_2)$,

$$\mathcal{F}_1\otimes\mathcal{F}_2 := \sigma(\mathcal{F}_1\times\mathcal{F}_2) = \sigma(A_1\times A_2 : A_1\in\mathcal{F}_1, A_2\in\mathcal{F}_2).$$

By induction, larger products are

$$\bigotimes_{j=1}^d \mathcal{F}_j = \sigma\left(\prod_{j=1}^d B_j : B_j\in\mathcal{F}_j \quad 1\le j\le d\right\}$$

$$= \sigma\left\{\prod_{j=1}^d A_j : A_j\in\mathcal{E}_j \quad 1\le j\le d\right\}$$

[HW]

$$\Omega_j\in\mathcal{E}_j,\quad \sigma(\mathcal{E}_j)=\mathcal{F}_j.$$

Thus $\mathcal{B}(\mathbb{R}^d) = \bigotimes_{j=1}^d \mathcal{B}(\mathbb{R})$.

Lemma: Let $\pi_k : \prod_{j=1}^d \Omega_j \to \Omega_k$ be the standard projection: $\pi_k(\omega_1,\omega_2,\dots,\omega_d)=\omega_k$.

Then $\bigotimes_{j=1}^d \mathcal{F}_j = \sigma\{\pi_k : 1\le k\le d\}$. [HW]

Lemma: (Product Measurability)

Let $(\Omega_j, \mathcal{F}_j)_{j\in J}$ and $(\Upsilon, \mathcal{B})$ be measurable spaces.

Then $f: \Upsilon \to \prod_{j\in J} \Omega_j$ is $\mathcal{B}/\bigotimes_{j\in J}\mathcal{F}_j$ - measurable

iff $\pi_k \circ f : \Upsilon \to \Omega_k$ is $\mathcal{B}/\mathcal{F}_k$ - measurable $\forall k \in J$.

Pf. We use the fact that $\bigotimes_{j\in J}\mathcal{F}_j = \sigma\{\pi_j : j\in J\} =: \mathcal{F}$

($\Rightarrow$) $\checkmark$ f is $\mathcal{B}/\mathcal{F}$ - meas, π_k is $\mathcal{F}/\mathcal{F}_k$ - meas.

$\therefore$ $\pi_k \circ f$ is $\mathcal{B}/\mathcal{F}_k$ - meas. (compositions)

($\Leftarrow$) $\mathcal{F} = \sigma\{\pi_j : j \in J\} = \sigma\left(\bigcup_{j\in J} \pi_j^* \mathcal{F}_j\right)$

$\therefore$ Suffices: $f^{-1}(A) \in \mathcal{B}$ $\forall A \in \bigcup_{j\in J} \pi_j^* \mathcal{F}_j$

$A = \pi_j^{-1}(E_j)$ $E_j \in \mathcal{F}_j$

$$f^{-1}(A) = \underbrace{f^{-1}(\pi_j^{-1}(E_j))}_{= (\pi_j \circ f)^{-1}(E_j)} \in \mathcal{B}.$$

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E.g. Let $f_j : \Omega_j \to \mathbb{R}$ be measurable functions. Then

$$f_1 \otimes f_2 : \Omega_1 \times \Omega_2 \to \mathbb{R} \quad \text{is defined by}$$

$$(f_1 \otimes f_2)(\omega_1, \omega_2) := f_1(\omega_1) \cdot f_2(\omega_2)$$

It is $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$ -measurable. ✓

$$f_1 \otimes f_2 = (f_1 \circ \pi_1) \cdot (f_2 \circ \pi_2)$$

We will use such "tensor product" functions frequently in our construction of product measure.

If $(\Omega_1, \mathcal{F}_1, \mu_1)$, $(\Omega_2, \mathcal{F}_2, \mu_2)$ are measure spaces, we want to construct a measure $\mu_1 \otimes \mu_2$ on $\mathcal{F}_1 \otimes \mathcal{F}_2$ satisfying

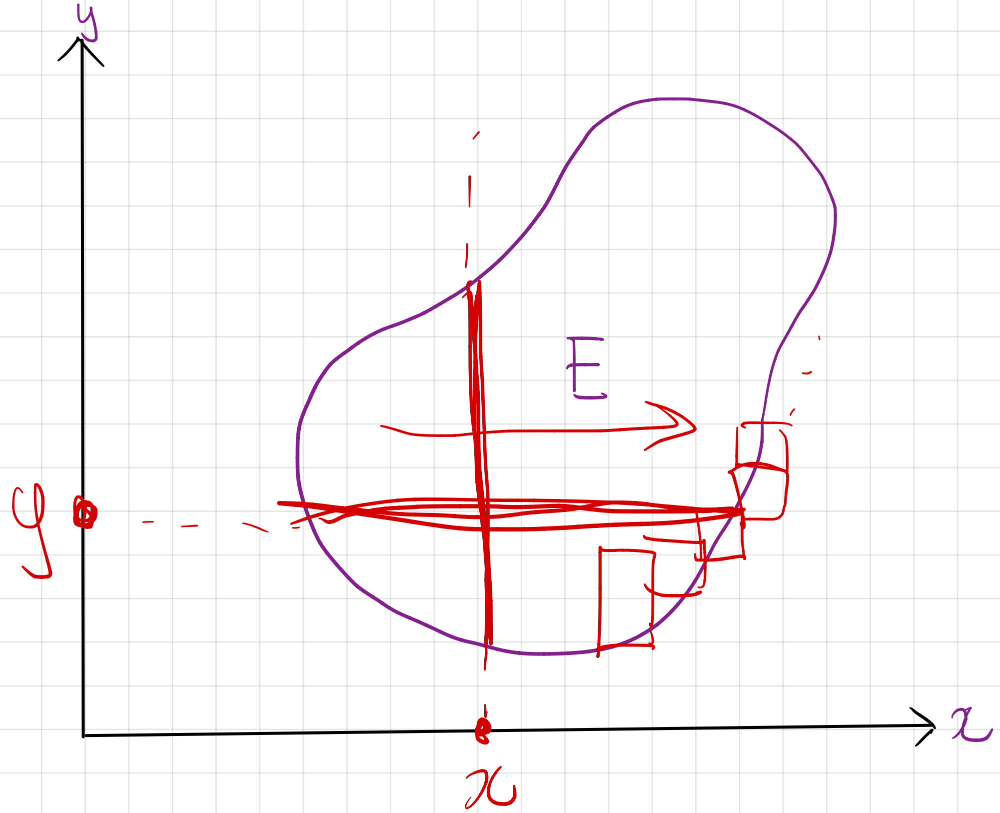
$$(\mu_1 \otimes \mu_2)(B_1 \times B_2) = \mu_1(B_1)\,\mu_2(B_2) \quad \text{for } B_j \in \mathcal{F}_j$$

$$\int_{\Omega_1 \times \Omega_2} f_1 \otimes f_2 \, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} f_1 \, d\mu_1 \cdot \int_{\Omega_2} f_2 \, d\mu_2 .$$

$f_1 = \mathbb{1}_{B_1}$

$f_2 = \mathbb{1}_{B_2}$

$\Rightarrow f_1 \otimes f_2 = \mathbb{1}_{B_1 \times B_2}$.



$$\text{Area}(E) = \iint \mathbb{1}_E \, dx\,dy$$

$$= \int \left(\int \mathbb{1}_E(x,y)\, dy \right) dx,$$

$$= \int \left(\int \mathbb{1}_E(x,y)\, dx \right) dy$$

To really understand product measure, we need to figure out how to do

Iterated Lebesgue integrals.

Theorem: [13.3] Let $(\Omega_j, \mathcal{F}_j, \mu_j)$ $j=1,2$ be σ-finite measure spaces. Let $f: \Omega_1 \times \Omega_2 \to [0,\infty)$ be a non-negative $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$-measurable function. Then:

1. (a) $\omega_1 \mapsto f(\omega_1, \omega_2)$ is $\mathcal{F}_1 / \mathcal{B}(\mathbb{R})$-measurable $\forall \omega_2 \in \Omega_2$

   (b) $\omega_2 \mapsto f(\omega_1, \omega_2)$ is $\mathcal{F}_2 / \mathcal{B}(\mathbb{R})$-measurable $\forall \omega_1 \in \Omega_1$

2. (a) $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2)$ is $\mathcal{F}_1 / \mathcal{B}(\overline{\mathbb{R}})$-measurable.

   (b) $\omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2)\, \mu_1(d\omega_1)$ is $\mathcal{F}_2 / \mathcal{B}(\overline{\mathbb{R}})$-measurable.

3. $$\int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1) = \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2)\, \mu_1(d\omega_1) \right) \mu_2(d\omega_2)$$

We'll prove this for μ_1, μ_2 finite measures; the extension to σ-finite is standard and boring.

Pf. <u>Step 1.</u> Verify that 1-3 hold for $f = f_1 \otimes f_2$, $f_j \in B(\Omega_j, \mathcal{F}_j)$

1. (a) $\omega_1 \mapsto f(\omega_1, \omega_2) = f_1(\omega_1) \cdot \underline{\underline{f_2(\omega_2)}} = \text{Const} \cdot f_1(\omega_1)$ ✓

   (b) $\omega_2 \mapsto f(\omega_1, \omega_2)$ analogous ✓

2. (a) $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) = \int_{\Omega_2} \underline{\underline{f_1(\omega_1)}}\, f_2(\omega_2)\, \mu_2(d\omega_2)$

   $= \underline{f_1(\omega_1)} \underbrace{\int_{\Omega_2} f_2\, d\mu_2}_{\text{Const.}}$ meas.

   (b) analogous ✓

3. $\int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$

   $\int_{\Omega_1} f_1(\omega_1) \underbrace{\int f_2\, d\mu_2}\, \mu_1(d\omega_1) = \int f_2\, d\mu_2 \cdot \int f_1\, d\mu_1$ ✓

Step 2. Let $\mathbb{H} = \{ f \in \mathbb{B}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) : 1\text{-}3 \text{ hold} \}$

Let $\mathbb{M} = \mathbb{B}(\Omega_1, \mathcal{F}_1) \otimes \mathbb{B}(\Omega_2, \mathcal{F}_2)$

- $\mathbb{M}$ is a multiplicative system:

$$(f_1 \otimes f_2) \cdot (g_1 \otimes g_2)(\omega_1, \omega_2) = (f_1(\omega_1) f_2(\omega_2)) \cdot (g_1(\omega_1) g_2(\omega_2))$$

$$= f_1(\omega_1) g_1(\omega_1) \cdot f_2(\omega_2) g_2(\omega_2) = (f_1 g_1 \otimes f_2 g_2)(\omega_1, \omega_2).$$

- $\sigma(\mathbb{M}) = \underline{\mathcal{F}_1 \otimes \mathcal{F}_2}$ ✓ $\quad \mathbb{M} \ni \mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2} = \mathbb{1}_{B_1 \times B_2} \quad \forall B_1 \in \mathcal{F}_1, B_2 \in \mathcal{F}_2$

- $\mathbb{H} \supseteq \mathbb{M} \cup \{1\} = \mathbb{M}$. ✓ Step 1

- $\mathbb{H}$ is closed under bounded convergence. ✓

Thus, by Dynkin's Multiplicative Systems Theorem,

$$\therefore \mathbb{H} \supseteq \mathbb{B}(\Omega_1 \times \Omega_2, \sigma(\mathbb{M})) = \mathbb{B}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2).$$

<u>Step 3</u>. Let $f \geq 0$ be $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}(\mathbb{R})$ - measurable.

For $n \in \mathbb{N}$, set $f_n = \min\{f, n\} \in \mathbb{B}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$

$f_n \uparrow f$ b/c f is $\mathbb{R}$-valued

Cor: There is a unique measure $\mu_1 \otimes \mu_2$ (μ_1, μ_2 σ-finite)
on $\mathcal{F}_1 \otimes \mathcal{F}_2$ s.t. $\mu_1 \otimes \mu_2 (E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$
for all $E_j \in \mathcal{F}_j$, and it is given by

$$(\mu_1 \otimes \mu_2)(E) = \int_{\Omega_1} \left(\int_{\Omega_2} \mathbb{1}_E(\omega_1, \omega_2)\, \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$$

$$= \int_{\Omega_2} \left(\int_{\Omega_1} \mathbb{1}_E(\omega_1, \omega_2)\, \mu_1(d\omega_1) \right) \mu_2(d\omega_2)$$

$\forall E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

$\therefore$ $\mu_1 \otimes \mu_2$ is σ-finite.

for $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

Pf. $\mu_1 \otimes \mu_2$ is finitely additive: $\mathbb{1}_{E_1 \sqcup E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2}$, $\int$'s are linear.

- If $E_n \uparrow E \Rightarrow \mathbb{1}_{E_n} \uparrow \mathbb{1}_E$ $\therefore$ $\mu_1 \otimes \mu_2(E_n) \uparrow \mu_1 \otimes \mu_2(E)$ by MCT.

$$\mu_1 \otimes \mu_2(E_1 \times E_2) = \int_{\Omega_1} \int_{\Omega_2} \underbrace{\mathbb{1}_{E_1 \times E_2}}_{\mathbb{1}_{E_1} \otimes \mathbb{1}_{E_2}} d\mu_2\, d\mu_1 = \int \mathbb{1}_{E_1} d\mu_1 \cdot \int \mathbb{1}_{E_2} d\mu_2$$

$$= \mu_1(E_1)\mu_2(E_2).$$

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