Bounded Convergence [Driver, §12.1]

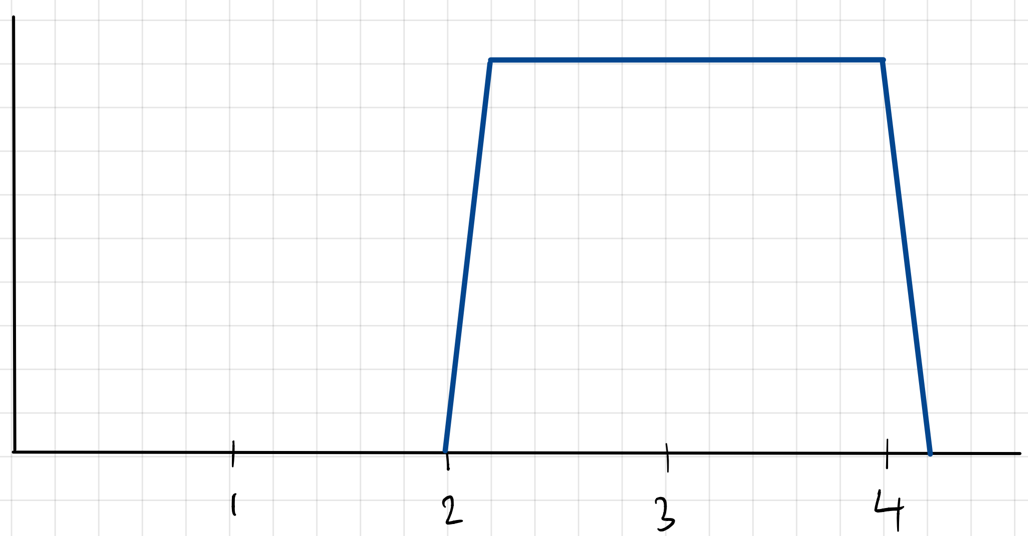
A set $\mathbb{H}$ of $\mathbb{R}$-valued functions on Ω is **closed under bounded convergence** if

$$f_n \in \mathbb{H},\ \exists M < \infty \text{ s.t. } |f_n(\omega)| \leq M \ \forall n \in \mathbb{N}, \omega \in \Omega$$
$$\& \quad \lim_{n\to\infty} f_n(\omega) = f(\omega) \in \mathbb{R} \ \forall \omega \in \Omega \quad \Rightarrow f \in \mathbb{H}.$$

Notation: $\mathcal{B}(\Omega, \mathcal{F}) := \{\text{bounded } \mathcal{F}/\mathcal{B}(\mathbb{R})\text{-measurable functions}\}$

$$\mathcal{B}(\Omega) := \mathcal{B}(\Omega, 2^{\Omega})$$

E.g. $C_c(\mathbb{R}), C_b(\mathbb{R})$ are **not** closed under bounded convergence



$$\varphi_n(x) = \begin{cases} 0, & x \leq a \text{ or } \geq b + \frac{1}{n} \\ n(x-a), & a \leq x \leq a + \frac{1}{n} \\ 1, & a + \frac{1}{n} \leq x \leq b \\ 1 - n(b-x), & b \leq x \leq b + \frac{1}{n} \end{cases}$$

Notation: Given a collection $\mathbb{M}$ of $\mathbb{R}$-valued bounded functions on Ω, let

$$\mathbb{H}(\mathbb{M}) := \text{the smalles subspace of } \mathbb{B}(\Omega) \text{ containing } \mathbb{M} \cup \{1\}, \text{ and closed under bounded convergence.}$$

Theorem: [12.5] (Dynkins Multiplicative Systems Theorem)

Let $\mathbb{H} \subseteq \mathbb{B}(\Omega)$ be a subspace, containing 1, and closed under bounded convergence.

Let $\mathbb{M} \subseteq \mathbb{H}$ be a multiplicative system: $f, g \in \mathbb{M} \Rightarrow f \cdot g \in \mathbb{M}$.

Then $\mathbb{H}$ contains all bounded $\sigma(\mathbb{M})$-measurable functions:

$$\mathbb{B}(\Omega, \sigma(\mathbb{M})) \subseteq \mathbb{H}.$$

In fact,

$$\mathbb{B}(\Omega, \sigma(\mathbb{M})) = \mathbb{H}(\mathbb{M}).$$

Cor: $\mathbb{H}(C_c(\mathbb{R})) = \mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. I.e. the bounded convergence closure of the compactly-supported continuous functions is all bounded Borel measurable functions.

Pf.

Cor: Suppose ν, μ are Borel probability measures on $\mathbb{R}$, and

$$\int_{\mathbb{R}} f \, d\mu = \int_{\mathbb{R}} f \, d\nu \qquad \forall f \in C_c(\mathbb{R})$$

Then $\mu = \nu$.

Pf.

Proof of Dynkin's Multiplicative Systems Theorem

We will prove that $\mathbb{H}(\mathbb{M}) = \mathbb{B}(\Omega, \sigma(\mathbb{M}))$. WLOG: $\mathbb{H} = \mathbb{H}(\mathbb{M})$.

Step 1: $\boxed{\mathbb{H} \text{ is an algebra of functions.}}$

We already know $\mathbb{H}$ is a subspace; need to show it is a multiplicative system.

Fix $f \in \mathbb{H}$, and define $\mathbb{H}^f := \{g \in \mathbb{H} : fg \in \mathbb{H}\}$. Must show $\mathbb{H}^f = \mathbb{H}$.

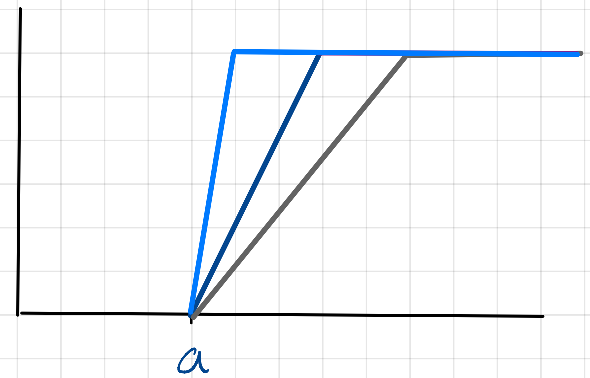
Step 2. $\boxed{\mathcal{F} := \{A \subseteq \Omega : \mathbb{1}_A \in \mathbb{H}\} \text{ is a } \sigma\text{-field.}}$

Step 3. $\boxed{\mathbb{B}(\Omega, \mathcal{F}) \subseteq \mathbb{H}.}$

Step 4. $\boxed{\sigma(M) \subseteq \mathcal{J}.}$

$$\sigma(M) = \sigma\left(\bigcup \{ f^{*}\mathcal{B}(\mathbb{R}) : f \in M \} \right) = \sigma\left(\bigcup \{ f^{-1}(a,\infty) : f \in M,\ a \in \mathbb{R} \} \right)$$

$\therefore$ Suffices to show $\{f > a\} \in \mathcal{J}$, i.e. $\mathbb{1}_{\{f>a\}} \in \mathbb{H}$ $\forall f \in M,\ a \in \mathbb{R}$.



Step 5. $\boxed{\mathbb{H}(\mathbb{M}) = \mathbb{B}(\Omega, \sigma(\mathbb{M}))}$.

By Step 4, $\sigma(\mathbb{M}) \subseteq \mathcal{F}$, $\therefore$ $\mathbb{B}(\Omega, \sigma(\mathbb{M})) \subseteq \mathbb{B}(\Omega, \mathcal{F})$.