Bounded Convergence [Driver, §12.1]

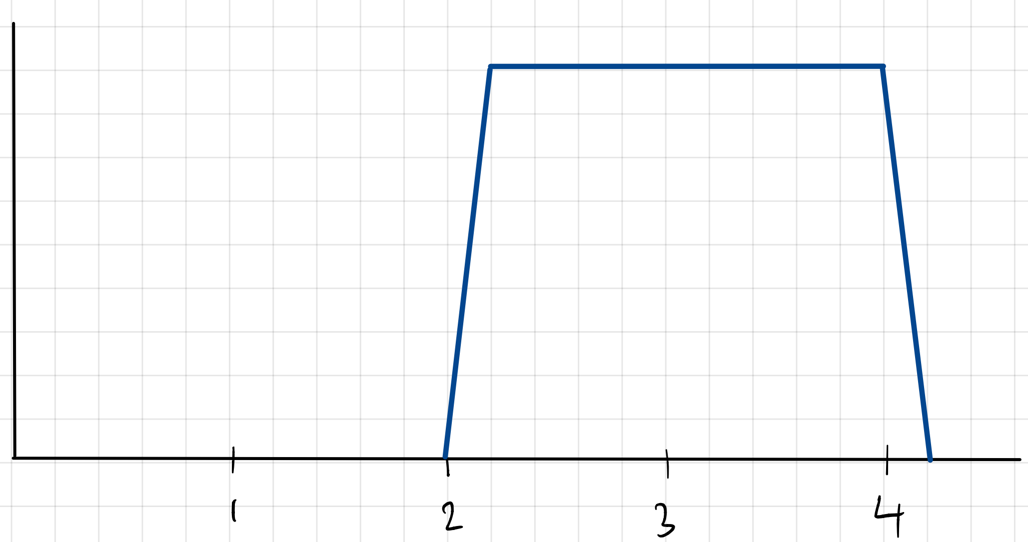
A set $\mathbb{H}$ of $\mathbb{R}$-valued functions on Ω is **closed under bounded convergence** if

$$f_n \in \mathbb{H},\ \exists M<\infty \text{ s.t. } |f_n(\omega)| \le M \quad \forall n\in\mathbb{N}, \omega\in\Omega$$
$$\& \quad \lim_{n\to\infty} f_n(\omega) = f(\omega) \in \mathbb{R} \quad \forall \omega \in \Omega \quad \Rightarrow f\in\mathbb{H}.$$

Notation: $\mathbb{B}(\Omega,\mathcal{F}) := \{\text{bounded } \mathcal{F}/\mathcal{B}(\mathbb{R})\text{-measurable functions}\}$

$\mathbb{B}(\Omega) := \mathbb{B}(\Omega, 2^{\Omega})$ } closed under bdd convergence.

E.g. $C_c(\mathbb{R}), C_b(\mathbb{R})$ are **not** closed under bounded convergence



$$\varphi_n(x) = \begin{cases} 0, & x\le a \text{ or } \ge b+\frac{1}{n} \\ n(x-a), & a\le x\le a+\frac{1}{n} \\ 1, & a+\frac{1}{n}\le x\le b \\ 1-n(b-x), & b\le x\le b+\frac{1}{n} \end{cases} \in C_c(\mathbb{R})$$

$\varphi_n \longrightarrow \mathbb{1}_{(a,b]} \notin C(\mathbb{R})$

$|\varphi_n| \le 1.$

Notation: Given a collection $\mathbb{M}$ of $\mathbb{R}$-valued bounded functions on Ω, let

$\mathbb{H}(\mathbb{M}) :=$ the smalles subspace of $\mathbb{B}(\Omega)$ containing $\mathbb{M} \cup \{1\}$, and closed under bounded convergence.

Theorem: [12.5] (Dynkins Multiplicative Systems Theorem)

Let $\mathbb{H} \subseteq \mathbb{B}(\Omega)$ be a subspace, containing 1, and closed under bounded convergence.

Let $\mathbb{M} \subseteq \mathbb{H}$ be a multiplicative system: $f, g \in \mathbb{M} \Rightarrow f \cdot g \in \mathbb{M}$.

Then $\mathbb{H}$ contains all bounded $\sigma(\mathbb{M})$-measurable functions:

$$\mathbb{B}(\Omega, \sigma(\mathbb{M})) \subseteq \mathbb{H}.$$

In fact, $\quad \mathbb{B}(\Omega, \sigma(\mathbb{M})) = \mathbb{H}(\mathbb{M}).$

Cor: $\mathbb{H}(C_c(\mathbb{R})) = \mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. I.e. the bounded convergence closure of the compactly-supported continuous functions is all bounded Borel measurable functions.

Pf. $\exists\ \varphi_n \in C_c(\mathbb{R})$ s.t. $\varphi_n \to \mathbb{1}_{(a,b]}$ (unif. bdd)

$\uparrow$

$\sigma(C_c(\mathbb{R}))/\mathcal{B}(\mathbb{R})$-meas. $\Rightarrow$ $\mathbb{1}_{(a,b]}$ is $\sigma(C_c(\mathbb{R}))/\mathcal{B}(\mathbb{R})$-meas.

$\therefore\ \underbrace{\mathbb{1}_{(a,b]}^{-1}\{1\}}_{=(a,b]} \in \sigma(C_c(\mathbb{R}))$

$\therefore\ \sigma(C_c(\mathbb{R})) \supseteq \mathcal{B}(\mathbb{R})$

$\downarrow$

$C_c(\mathbb{R})$ is a mult. system

Also $C_c(\mathbb{R})$ are Borel-measurable

$\therefore\ \sigma(C_c(\mathbb{R})) \subseteq \mathcal{B}(\mathbb{R})$.

$\therefore\ \mathcal{B}(\mathbb{R}) = \sigma(C_c(\mathbb{R}))$

$$\therefore\ \mathbb{H}(C_c(\mathbb{R})) = \mathbb{B}(\mathbb{R}, \sigma(C_c(\mathbb{R}))) = \mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

Cor: Suppose ν, μ are Borel probability measures on $\mathbb{R}$, and

$$\int_{\mathbb{R}} f\,d\mu = \int_{\mathbb{R}} f\,d\nu \qquad \forall f \in C_c(\mathbb{R})$$

Then $\mu = \nu$.

Pf. $C_c(\mathbb{R})$ form a mult. system in $\mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$\mathbb{H} := \left\{ g \in \mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R})) : \int g\,d\mu = \int g\,d\nu \right\} \supseteq C_c(\mathbb{R}) \supseteq \{1\}$$

closed under bdd conv. by DCT.

$\therefore$ by Dynkin's thm, $\mathbb{H} \supseteq \mathbb{B}(\mathbb{R}, \sigma(C_c(\mathbb{R})))$
$= \mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\therefore$ if $B \in \mathcal{B}(\mathbb{R})$, $\underset{\mu(B)}{\underset{\parallel}{\int 1_B\,d\mu}} = \underset{\nu(B)}{\underset{\parallel}{\int 1_B\,d\nu}}$

Proof of Dynkin's Multiplicative Systems Theorem

We will prove that $\mathbb{H}(\mathbb{M}) = \mathbb{B}(\Omega, \sigma(\mathbb{M}))$. WLOG: $\mathbb{H} = \mathbb{H}(\mathbb{M})$.

$\subseteq \mathbb{H}$

✓

Step 1: $\boxed{\mathbb{H} \text{ is an algebra of functions.}}$

We already know $\mathbb{H}$ is a subspace; need to show it is a multiplicative system.

Fix $f \in \mathbb{H}$, and define $\mathbb{H}^f := \{g \in \mathbb{H} : fg \in \mathbb{H}\}$. Must show $\mathbb{H}^f = \mathbb{H}$.

- $\mathbb{H}^f \subseteq \mathbb{H}$ is a subspace
- $\mathbb{1} \in \mathbb{H}^f$
- $\mathbb{H}^f$ is closed under bdd convergence.

$$\underset{\mathbb{H}^f}{\overset{}{g_n}} \xrightarrow{\text{unif.}} g \;\therefore\; \in \mathbb{H} \qquad \underset{\mathbb{H}}{g_n f} \longrightarrow gf \;\therefore\; \in \mathbb{H}. \Rightarrow g \in \mathbb{H}^f.$$

Rinse, lather, repeat.

If $f \in \mathbb{M}$, $\therefore \mathbb{M} \subseteq \mathbb{H}^f \;\therefore\; \underset{\mathbb{H}}{\mathbb{H}(\mathbb{M})} \subseteq \mathbb{H}^f \subseteq \mathbb{H}$

$\therefore \mathbb{H} = \mathbb{H}^f$. $\left.\begin{array}{l}\text{if } f \in \mathbb{M},\\ \& \; g \in \mathbb{H}\end{array}\right\} \Rightarrow fg \in \mathbb{H}.$

Step 2. $\boxed{\mathcal{F} := \{A \subseteq \Omega : \mathbb{1}_A \in \mathbb{H}\} \text{ is a } \sigma\text{-field.}}$ ✓

$\mathbb{1}_\phi = 0 \in \mathbb{H}$ ✓

$\mathbb{1}_\Omega = 1 \in \mathbb{H}$

If $A \in \mathcal{F}$, $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A \quad \therefore A^c \in \mathcal{F}$.

$\Downarrow \qquad \therefore \Uparrow$

$\mathbb{1}_A \in \mathbb{H} \qquad \mathbb{H}$

$\therefore A \cap B \in \mathcal{F}$.

$\Uparrow$

If $A, B \in \mathcal{F}$

$\mathbb{1}_A, \mathbb{1}_B \in \mathbb{H} \Rightarrow \mathbb{H} \ni \mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A \cap B}$

If $A_n \in \mathcal{F}$, $A_n \uparrow A$ Then $\mathbb{1}_{A_n} \xrightarrow{\text{bded}} \mathbb{1}_A \overset{\therefore}{\in} \mathbb{H} \Rightarrow A \in \mathcal{F}$.

$\Uparrow$

$\mathbb{H}$

Step 3. $\boxed{\mathbb{B}(\Omega, \mathcal{F}) \subseteq \mathbb{H}}$. ✓

If $A \in \mathcal{F}$, $\mathbb{1}_A \in \mathbb{H}$. $\Rightarrow$ $\{\mathcal{F}\text{-meas. simple functions}\} \subseteq \mathbb{H}$.

Fix $f \in \mathbb{B}(\Omega, \mathcal{F})$. Find $\underset{\substack{\Downarrow\\ \varphi_n^+ - \varphi_n^-}}{\overset{\Downarrow}{\varphi_n}} \xrightarrow{\text{bdded}} f \; \therefore \in \mathbb{H}$.

$\varphi_n^+ \uparrow f_+$

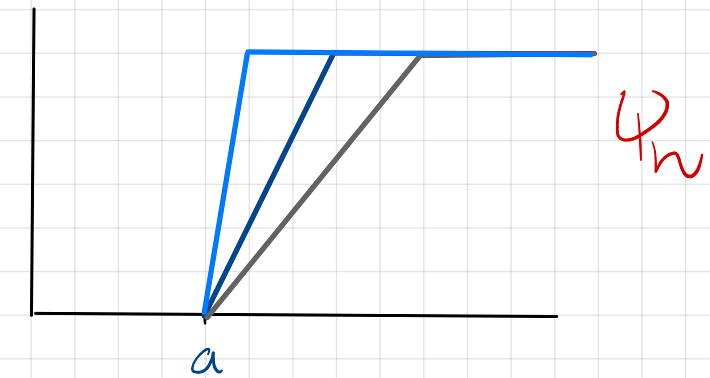
$\varphi_n^- \uparrow f_-$

Step 4. $\boxed{\sigma(\mathbb{M}) \subseteq \mathcal{J}}$ $= \{A \subseteq \Omega : \mathbb{1}_A \in \mathbb{H}\}$

$\sigma(\mathbb{M}) = \sigma\left(\bigcup\{f^{*}\mathcal{B}(\mathbb{R}) : f \in \mathbb{M}\}\right) = \sigma\left(\bigcup\{f^{-1}(a,\infty) : f \in \mathbb{M}, a \in \mathbb{R}\}\right)$

$\therefore$ Suffices to show $\{f > a\} \in \mathcal{J}$, i.e. $\mathbb{1}_{\{f>a\}} \in \mathbb{H}$ $\forall f \in \mathbb{M}, a \in \mathbb{R}$.

$\mathbb{1}_{(a,\infty)}$ $\leftarrow$ approximate by ψ_n



$\therefore$ $\psi_n \circ f \xrightarrow{\text{bdd el.}} \mathbb{1}_{\{f>a\}}$ $\therefore \in \mathbb{H}$

$\uparrow$

$f \in \mathbb{M} \subseteq B(\Omega)$, so $\sup|f| = M < \infty$.

By Weierstrass approx., $\exists\, p_{n,k} \xrightarrow[k\to\infty]{\text{unif.}} \psi_n$ on $[-M, M]$.

Since $\mathbb{H}$ is an algebra, $p_{n,k} \circ f \in \mathbb{H}$ $\forall n, k$.

$p_{n,k} \circ f \xrightarrow[k\to\infty]{\text{bdd el}} \psi_n \circ f$

$\mathbb{H}$ (∋ $p_{n,k}\circ f$) $\therefore$ $\psi_n \circ f \in \mathbb{H}$

Step 5. $\boxed{\mathbb{H}(\mathbb{M}) = \mathcal{B}(\Omega, \sigma(\mathbb{M}))}$.

By Step 4, $\sigma(\mathbb{M}) \subseteq \mathcal{F}$, $\therefore$ $\mathcal{B}(\Omega, \sigma(\mathbb{M})) \subseteq \mathcal{B}(\Omega, \mathcal{F})$.

$\subseteq \mathbb{H}(\mathbb{M})$ by Step 3.

$$\mathbb{M} \subseteq \underset{\substack{\cup\\ 1}}{\mathcal{B}(\Omega, \sigma(\mathbb{M}))} \subset \mathcal{B}(\Omega)$$

$\therefore$ by Def $\mathbb{H}(\mathbb{M}) \subseteq \mathcal{B}(\Omega, \sigma(\mathbb{M}))$.

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