Remember L^p:

$$L^p(\Omega, \mathcal{F}, \mu) = \{ f \in L^0(\Omega, \mathcal{F}, \mu) : \int |f|^p d\mu < \infty \}$$

We've looked carefully at L^1 and L^2. More generally:

Theorem: [17.24] For $1 \leq p < \infty$,

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p d\mu \right)^{1/p}$$

defines a norm on $L^p(\Omega, \mathcal{F}, \mu)$. In particular:

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

for $f, g \in L^p$. Thus L^p is a normed vector space.

This is Minkowski's inequality. We proved it already for $p=1,2$.
In general, it goes beyond the scope of what we need to prove it.

L^p vs. L^0 Convergence

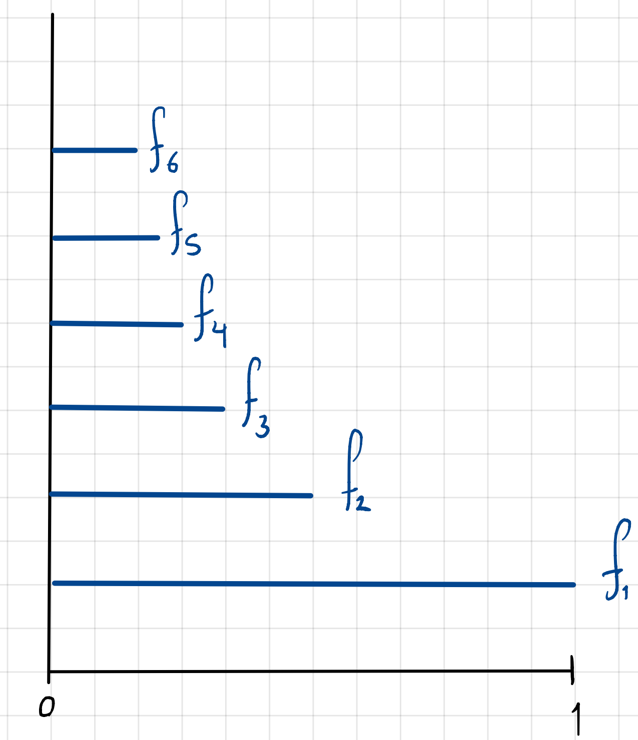
Lemma: Let $f_n, f \in L^p(\Omega, \mathcal{F}, \mu)$ $(1 \leq p < \infty)$ with $\|f_n - f\|_{L^p} \to 0$.
Then $f_n \to_\mu f$.

Pf.

The converse is false.

Eg. $(\Omega, \mathcal{F}, \mu) = ([0,1], \mathcal{B}[0,1], \lambda)$.

$$f_n = n \cdot \mathbb{1}_{[0, \frac{1}{n}]}$$



Theorem: [17.27] For $1 \le p < \infty$, L^p is Cauchy complete:

$f_n \in L^p$, $\|f_n - f_m\|_{L^p} \to 0$ as $n, m \to \infty$

$\Longrightarrow \exists f \in L^p$ s.t. $\|f_n - f\|_{L^p} \to 0$.

Pf.

Summary

L^0 convergence

$(p \geq 1)$ L^p convergence

a.s. convergence

Theorem: (L^0 - L^1 DCT)

Let $f_n, g_n, g \in L^1$, $f \in L^0$, s.t.

(1) $|f_n| \le g_n$ a.s.

(2) $f_n \to_\mu f$, $g_n \to_\mu g$

(3) $\int g_n \to \int g$

Then $f \in L^1$ and $\|f_n - f\|_{L^1} \to 0$. (In particular, $\int f_n \to \int f$.)

Pf.