Remember L^p:

$$L^p(\Omega, \mathcal{F}, \mu) = \{ f \in L^0(\Omega, \mathcal{F}, \mu) : \int |f|^p d\mu < \infty \} / \mu\text{-null sets}$$

We've looked carefully at L^1 and L^2. More generally:

Theorem: [17.24] For $1 \leq p < \infty$,

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p d\mu \right)^{1/p}$$

defines a norm on $L^p(\Omega, \mathcal{F}, \mu)$. In particular:

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

for $f, g \in L^p$. Thus L^p is a normed vector space.

This is Minkowski's inequality. We proved it already for $p = 1, 2$.
In general, it goes beyond the scope of what we need to prove it.

L^p vs. L^0 Convergence

Lemma: Let $f_n, f \in L^p(\Omega, \mathcal{F}, \mu)$ $(1 \le p < \infty)$ with $\|f_n - f\|_{L^p} \to 0$.
Then $f_n \to_\mu f$.

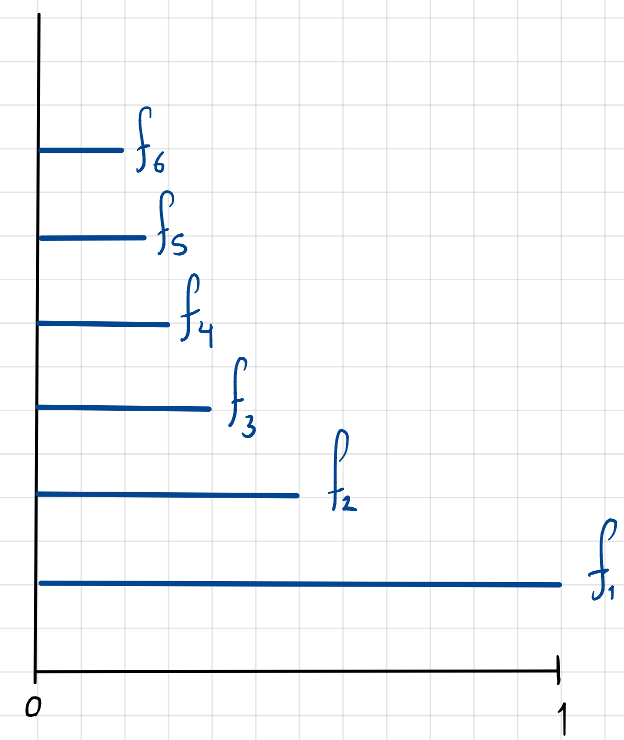
Pf. Markov's inequality: $\forall \varepsilon > 0$,

$$\mu\{|f_n - f| \ge \varepsilon\} \le \frac{1}{\varepsilon^p}\int |f_n - f|^p \, d\mu = \frac{1}{\varepsilon^p}\|f_n - f\|_{L^p}^p \to 0.$$

The converse is false.

Eg. $(\Omega, \mathcal{F}, \mu) = ([0,1], \mathcal{B}[0,1], \lambda)$.

$$f_n = n \cdot \mathbb{1}_{[0, \frac{1}{n}]}$$



$$\lambda\{|f_n| \ge \varepsilon\} \underset{n \ge 1}{\le} \lambda\{|f_n| \ge n\} = \frac{1}{n} \to 0.$$

$\therefore f_n \to_\lambda 0$.

$p \ge 1$

$$\|f_n - 0\|_{L^p}^p = \int |f_n|^p \, d\lambda = \int_0^{1/n} n^p \, d\lambda = \frac{1}{n} \cdot n^p = n^{p-1} \quad \therefore \|f_n - 0\|_{L^p} = n^{1 - \frac{1}{p}} \not\to 0$$

Theorem: [17.27] For $1 \le p < \infty$, L^p is Cauchy complete:

$f_n \in L^p$, $\|f_n - f_m\|_{L^p} \to 0$ as $n, m \to \infty$

$\Longrightarrow \exists f \in L^p$ s.t. $\|f_n - f\|_{L^p} \to 0$.

Pf. L^p - Cauchy $\Rightarrow L^0$ - Cauchy:

$$\mu\{|f_n - f_m| \ge \varepsilon\} \overset{\text{Markov}}{\le} \frac{1}{\varepsilon^p}\int |f_n - f_m|^p \, d\mu = \frac{\|f_n - f_m\|_{L^p}^p}{\varepsilon^p} \to 0 \quad n, m \to \infty.$$

$\therefore (f_n)$ is L^0 - Cauchy.

$\therefore \exists\, f_{n_k} \to f$ a.s., $f \in L^0$.

$$\|f_{n_k} - f\|_{L^p}^p \overset{\text{a.s.}}{=} \int \lim_{j\to\infty} |f_{n_k} - f_{n_j}|^p \, d\mu \overset{\text{Fatou}}{\le} \liminf_{j\to\infty} \int |f_{n_k} - f_{n_j}|^p \, d\mu = \liminf_{j\to\infty} \|f_{n_k} - f_{n_j}\|_{L^p}^p$$

$$\lim_{k\to\infty} \left(\quad \right) \le \lim_{k\to\infty} \liminf_{j\to\infty} \|f_{n_k} - f_{n_j}\|_{L^p}^p = 0.$$

$$\therefore \|f_n - f\|_{L^p} = \|(f_n - f_{n_k}) + (f_{n_k} - f)\|_{L^p} \le \underbrace{\|f_n - f_{n_k}\|_{L^p}}_{\to 0 \text{ as } n, k \to \infty} + \underbrace{\|f_{n_k} - f\|_{L^p}}_{\to 0} \to 0 \text{ as } k \to \infty$$

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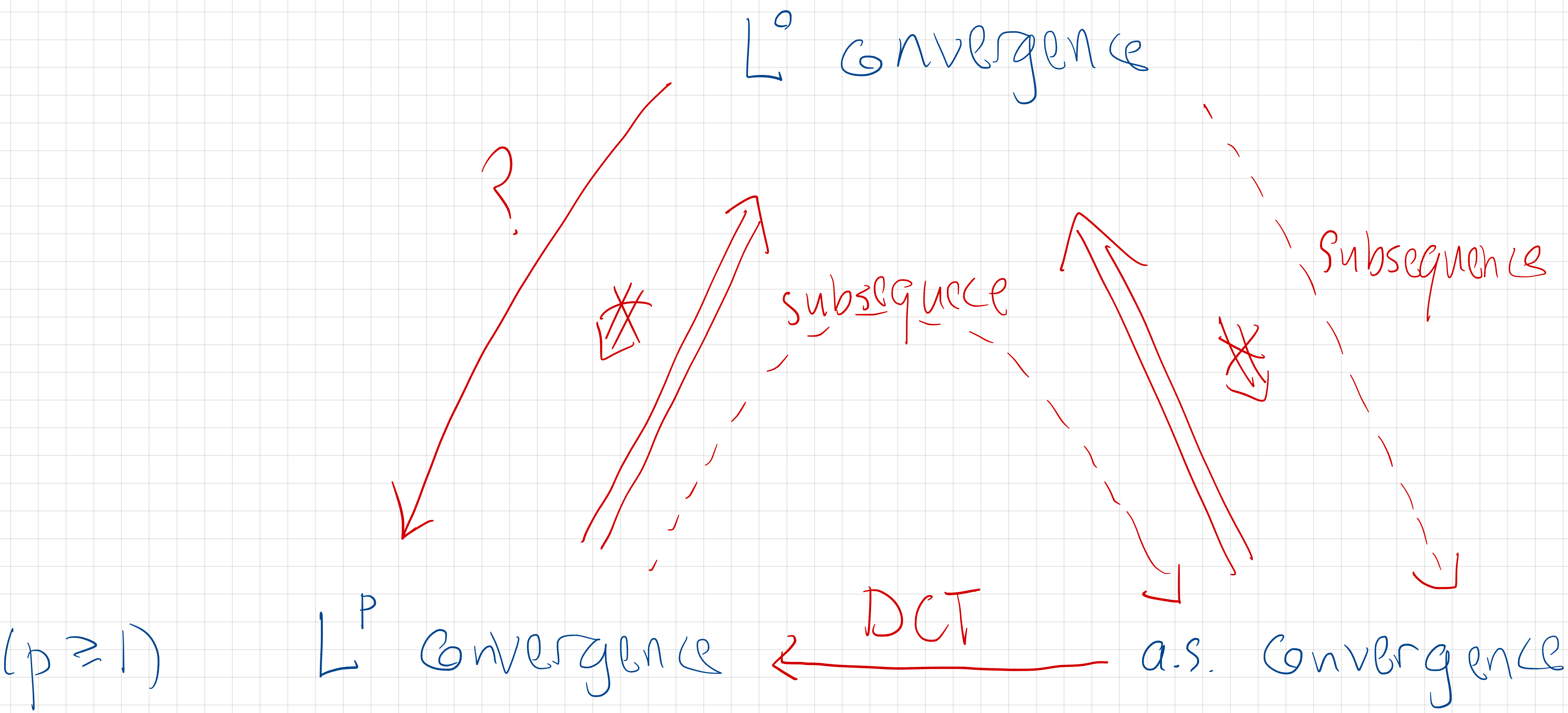
Summary



L^0 convergence
?
subsequence
Subsequence
DCT
$(p \geq 1)$
L^p convergence
a.s. convergence

Theorem: ($L^0 - L^1$ DCT)

Let $f_n, g_n, g \in L^1$, $f \in L^0$, s.t.

(1) $|f_n| \le g_n$ a.s.

(2) $f_n \to_\mu f$, $g_n \to_\mu g$

(3) $\int g_n \to \int g$

Then $f \in L^1$ and $\|f_n - f\|_{L^1} \to 0$. (In particular, $\int f_n \to \int f$.)

Pf. By (2), can find $f_{n_k} \to f$ & $g_{n_k} \to g$ a.s.

$$|f| = \lim_{k\to\infty} |f_{n_k}| \le \lim_{n\to\infty} g_{n_k} = g \quad \text{a.s.}$$

a.s.

S'pose that $\cancel{\|f_n - f\|_{L^1} \not\to 0}$. $\exists \varepsilon > 0$, & n_k s.t. $\|f_{n_k} - f\|_{L^1} \ge \varepsilon \ \forall k$,

But $f_{n_k} \to_\mu f$, $\therefore$ $f_{n_k} \to f$ a.s. $g_{n_k} \to g$ a.s.

$|f - f_{n_k}| \le |f| + |f_{n_k}| \le g + g_{n_k} \to 2g$. Also $\int (g + g_{n_k}) = \int g + \int g_{n_k} \to \int 2g$.

$\therefore$ by DCT $\Rightarrow \int |f - f_{n_k}| \to 0$. ↯ $\therefore$ $|\int f_n - \int f| = |\int (f_n - f)| \le \int |f_n - f| \to 0$. ///