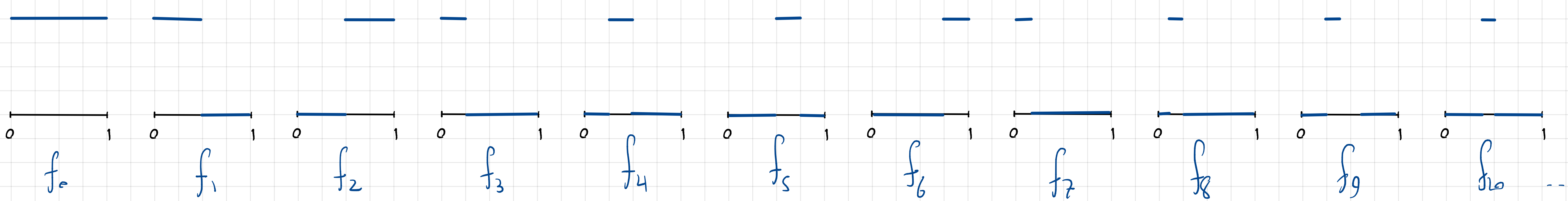
Convergence in Measure [Driver, §17.2]

Def: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.
Given measurable functions $f_n, f : \Omega \to \mathbb{R}$,
we say $f_n \xrightarrow{\mu} f$ (converges in measure)
if
$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mu\{|f_n - f| \geq \varepsilon\} = 0 .$$

If μ is a probability measure, we call this
convergence in probability.

Eg.



$$f_{2^0+2^1+\cdots+2^{n-1}+k} =$$

Usually, "convergence" is wrt a metric.

$d_\varepsilon(f,g) := \mu\{|f-g| \geq \varepsilon\}$ is <u>not</u> a metric.

It's not even a pseudometric. But...

$d_\varepsilon(f,h) = \mu\{|f-h| \geq \varepsilon\} = \mu\{|(f-g)+(g-h)| \geq \varepsilon\}$

Theorem: $d_0(f,g) := \qquad \mu\{|f-g| \geq \varepsilon\}$

defines a metric on the space of measurable functions and $f_n \to_\mu f$ iff $d_0(f_n, f) \to 0$.

Def: $L^0(\Omega, \mathcal{F}, \mu) := \{ f : \Omega \to \mathbb{R} \text{ measurable} \} / \mu\text{-null sets}$ equipped with the metric d_0.

In a probability space, there are other ways to metrize convergence in probability. E.g.

is a metric on $L^0(\Omega, \mathcal{F}, \mathbb{P})$, and $X_n \to_{\mathbb{P}} X$ iff $d(X_n, X) \to 0$. [HW]

In fact, L^0 is a complete metric space: Cauchy sequences converge. Let's write this without explicit reference to d_0.

Def: A sequence $f_n \in L^0$ is Cauchy in measure (aka L^0-Cauchy) if

$$\forall \varepsilon > 0 \qquad \lim_{n,m \to \infty} \mu\{ |f_n - f_m| \geq \varepsilon \} = 0 .$$

Theorem: [17.9] Let $f_n, g_n, f, g \in L^0(\Omega, \mathcal{F}, \mu)$.

1. (Uniqueness of limits) If $f_n \to_\mu f$ and $f_n \to_\mu g$, then $f = g$ a.s.$[\mu]$.

2. (Limits and vector space ops) If $\alpha, \beta \in \mathbb{R}$, $f_n \to_\mu f$, and $g_n \to_\mu g$, then $\alpha f_n + \beta g_n \to_\mu \alpha f + \beta g$.

3. If $f_n \to_\mu f$, then $\{f_n\}$ is Cauchy in measure.

Pf.

Theorem: [17.9] If $\{f_n\}$ is an L^0-Cauchy sequence, then $\exists f \in L^0$ s.t. some subsequence $f_{n_k} \to f$ a.s.

Moreover, $f_n \to_\mu f$.

Pf.

Now we must show that the full sequence $f_n \to_\mu f$.

Claim: for any $\ell \in \mathbb{N}$, $\mu\{|f - f_{n_{\ell+1}}| \geq 2^{-\ell}\} \leq 2^{-\ell}$.

We've seen that convergence in measure does not imply a.s. convergence; however, it does imply a.s. convergence of a subsequence.

In the converse direction, we have:

Theorem: [17.6] If $f_n \to f$ a.s. $[\mu]$, then $f_n \to_\mu f$.

Pf.