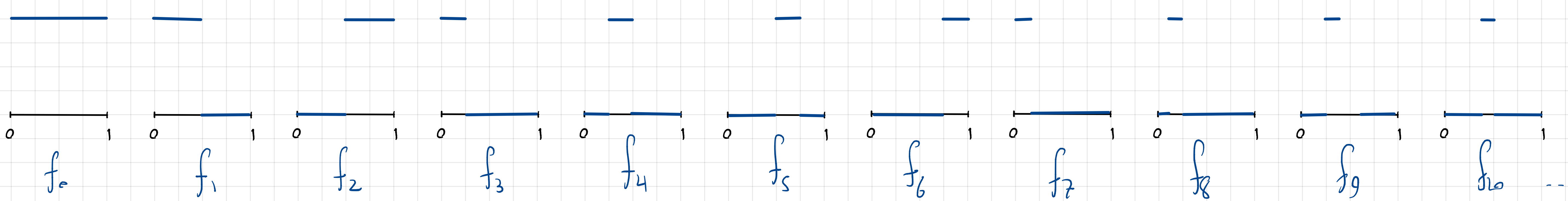
Convergence in Measure [Driver, §17.2]

Def: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Given measurable functions $f_n, f : \Omega \to \mathbb{R}$, we say $f_n \to_\mu f$ (converges in measure) if

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mu\{|f_n - f| \geq \varepsilon\} = 0.$$

If μ is a probability measure, we call this convergence in probability.

Eg.



$f_n \not\to 0$ a.s.

$\forall x \in [0,1]$, $f_n = 1$ on an interval containing x for arb. large n.

$$f_{\underset{k \in \{1,\dots,2^n\}}{2^0+2^1+\cdots+2^{n-1}+k}} = \mathbb{1}_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]} \quad \text{for } 1 \leq k \leq 2^n.$$

$$\lambda\{|f_{2^0+\cdots+2^{n-1}+k} - 0| \geq \varepsilon\} \leq \begin{cases} 0 & \varepsilon > 1 \\ 2^{-n} & 0 < \varepsilon \leq 1 \end{cases} \longrightarrow 0.$$

$\therefore f_n \to_\lambda 0.$

Usually, "convergence" is wrt a metric.

$d_\varepsilon(f,g) := \mu\{|f-g| \geq \varepsilon\}$ is <u>not</u> a metric.

It's not even a pseudometric. But...

$$d_\varepsilon(f,h) = \mu\{|f-h| \geq \varepsilon\} = \mu\{|(f-g)+(g-h)| \geq \varepsilon\}$$

$$\leq \mu\{|f-g|+|g-h| \geq \varepsilon\}$$

$$\leq \mu\left(\{|f-g| \geq \tfrac{\varepsilon}{2}\} \cup \{|g-h| \geq \tfrac{\varepsilon}{2}\}\right)$$

$$\leq \mu\{|f-g| \geq \tfrac{\varepsilon}{2}\} + \mu\{|g-h| \geq \tfrac{\varepsilon}{2}\}$$

$$= d_{\varepsilon/2}(f,g) + d_{\varepsilon/2}(g,h)$$

$\{\omega : |a(\omega)+b(\omega)| \geq \varepsilon\}$

$\subseteq \{\omega : |a(\omega)|+|b(\omega)| \geq \varepsilon\}$

If $|a|+|b| \geq \varepsilon$

$\Rightarrow |a| \geq \varepsilon/2$ or $|b| \geq \varepsilon/2$

Theorem: $d_0(f,g) := \inf_{\varepsilon>0} \left(\mu\{|f-g| \geq \varepsilon\} + \varepsilon\right)$ defines a metric on the space of measurable functions (mod. μ-null sets) and $f_n \xrightarrow{\mu} f$ iff $d_0(f_n,f) \to 0$.

Def: $L^0(\Omega, \mathcal{F}, \mu) := \{ f : \Omega \to \mathbb{R} \text{ measurable} \} / \mu\text{-null sets}$
equipped with the metric d_0.

/ In a probability space, there are other ways to metrize convergence in probability. E.g.

$$d(X, Y) := \mathbb{E}[\min(|X-Y|, 1)].$$

is a metric on $L^0(\Omega, \mathcal{F}, \mathbb{P})$, and $X_n \to_p X$ iff $d(X_n, X) \to 0$. [HW] /

In fact, L^0 is a complete metric space: Cauchy sequences converge. Let's write this without explicit reference to d_0.

Def: A sequence $f_n \in L^0$ is Cauchy in measure (aka L^0-Cauchy) if

$$\forall \varepsilon > 0 \quad \lim_{n,m \to \infty} \mu\{ |f_n - f_m| \geq \varepsilon \} = 0.$$

Theorem: [12.9] Let $f_n, g_n, f, g \in L^0(\Omega, \mathcal{F}, \mu)$.

1. (Uniqueness of limits) If $f_n \to_\mu f$ and $f_n \to_\mu g$, then $f = g$ a.s. $[\mu]$.

2. (Limits and vector space ops) If $\alpha, \beta \in \mathbb{R}$, $f_n \to_\mu f$, and $g_n \to_\mu g$, then $\alpha f_n + \beta g_n \to_\mu \alpha f + \beta g$.

3. If $f_n \to_\mu f$, then $\{f_n\}$ is Cauchy in measure.

Pf. 3. $|f_n - f_m| = |f_n - f + f - f_m| \leq |f_n - f| + |f_m - f|$

$$\mu\{|f_n - f_m| \geq \varepsilon\} \leq \underbrace{\mu\{|f_n - f| \geq \varepsilon/2\}}_{\xrightarrow{n \to \infty} 0} + \underbrace{\mu\{|f_m - f| \geq \varepsilon/2\}}_{\xrightarrow{m \to \infty} 0} \to 0 \text{ as } n, m \to \infty.$$

Theorem: [17.9] If $\{f_n\}$ is an L^0-Cauchy sequence, then $\exists f \in L^0$ s.t. some subsequence $f_{n_k} \to f$ a.s. ☆ NB!

Moreover, $f_n \to_\mu f$.

Pf. $\mu\{|f_n - f_m| \geq \varepsilon\} \to 0$ as $n, m \to \infty$. For each k, choose $n_{k+1} > n_k$ s.t. $\mu\{|f_{n_{k+1}} - f_{n_k}| \geq 2^{-k}\} \leq 2^{-k}$

$\therefore \sum_{k=1}^{\infty} \mu\{\text{''} \quad \text{''}\} < \infty$, by Borel-Cantelli (I)

$$\therefore \mu\{\underbrace{|f_{n_{k+1}} - f_{n_k}| \geq 2^{-k} \text{ i.o. } k}_{E}\} = 0.$$

$\therefore$ On E^c, $|f_{n_{k+1}} - f_{n_k}| \leq 2^{-k}$ for all large k.

Define $f = f_{n_1} + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$ on E^c ($f = 0$ on E)

$f_{n_{j+1}} = f_{n_1} + \sum_{k=1}^{j} (f_{n_{k+1}} - f_{n_k}) \longrightarrow f$ on E^c. $f_{n_j} \longrightarrow f \in L^0$ a.s.

Now we must show that the full sequence $f_n \to_\mu f$.

Claim: for any $\ell \in \mathbb{N}$, $\mu\{|f - f_{n_{\ell+1}}| \geq 2^{-\ell}\} \leq 2^{-\ell}$.

$$2^{-\ell} \leq \left| \sum_{k=\ell+1}^{\infty} (f_{n_{k+1}} - f_{n_k}) \right| \leq \sum_{k=\ell+1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

$$\sum_{k=\ell+1}^{\infty} 2^{-k} = 2^{-\ell}$$

$\exists\, k$ s.t. $|f_{n_{k+1}} - f_{n_k}| \geq 2^{-k}$

$\therefore$ by union bond

$$\therefore \mu\{|f - f_{n_{\ell+1}}| \geq 2^{-\ell}\} \leq \sum_{k=\ell+1}^{\infty} \mu\{|f_{n_{k+1}} - f_{n_k}| \geq 2^{-k}\}$$

$\leq 2^{-\ell}$ for large ℓ.

$$\mu\{|f - f_n| \geq \varepsilon\} \leq \mu\{|f - f_{n_\ell}| \geq \varepsilon/2\} + \mu\{|f_{n_\ell} - f_n| \geq \varepsilon/2\}$$

first term $\to 0$ as $\ell \to \infty$.

second term $\to 0$ as $n, \ell \to \infty$, by L^0-Cauchy.

$\therefore f_n \to_\mu f$. ///

We've seen that convergence in measure does not imply a.s. convergence; however, it does imply a.s. convergence of a subsequence.

In the converse direction, we have:

Theorem [17.6] If $f_n \to f$ a.s. $[\mu]$, then $f_n \to_\mu f$.

Pf. For any $\varepsilon > 0$, $\mu\{|f_n - f| \geq \varepsilon \text{ i.o.}\} = 0$. Let $A_n = \{|f_n - f| \geq \varepsilon\}$

$$0 = \mu\{A_n \text{ i.o.}\} = \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n\right)$$

$$= \lim_{k \to \infty} \mu\left(\bigcup_{n \geq k} A_n\right)$$

$$A_k \subseteq \bigcup_{n \geq k} A_n.$$

$$\therefore \mu(A_k) \leq \mu\left(\bigcup_{n \geq k} A_n\right) \to 0.$$

$$\mu(A_k) = \mu\{|f_k - f| \geq \varepsilon\}$$

$\therefore f_k \to_\mu f$. ///