Chebyshev's Inequality

Recall Markov's inequality: if $f \geq 0$, $\varepsilon, p > 0$, then

$$\mu\{f \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \int f^p$$

Suppose μ is a probabity measure, $X \in L^2$. Set $p = 2$, and apply Markov's inequality to $f = |\mathring{X}|$.

$$\mathbb{P}(|\mathring{X}| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} \mathbb{E}[|\mathring{X}|^2]$$

This is Chebyshev's inequality.

Alternative form: let $\sigma(X) = \sqrt{Var(X)}$ standard deviation

set $\varepsilon = k\sigma(X)$.

$$\therefore \mathbb{P}(|X - \mathbb{E}[X]| \geq k\sigma(X)) \leq$$

What is $\mathbb{E}$, Really?

Eg. Toss a fair coin n times.

$$\Omega = \{\omega = (\omega_1, \dots, \omega_n),\ \omega_j \in \{0,1\}\}$$

$$\mathcal{F} = 2^\Omega \qquad \mathbb{P}\{\omega\} = 2^{-n} \text{ for all } \omega \in \Omega.$$

$$X_j(\omega) = \omega_j$$

How many Heads come up?

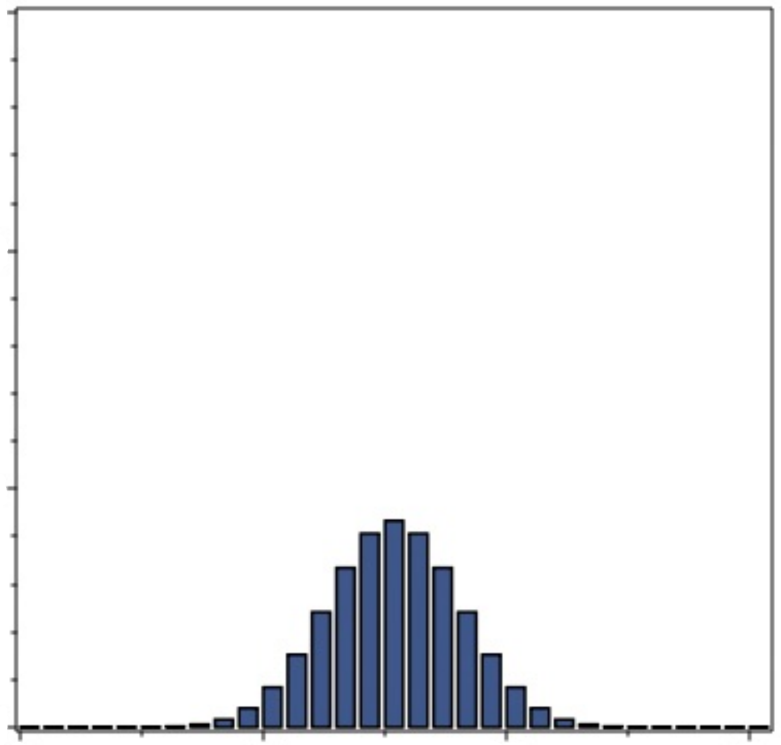
$$S_n = X_1 + X_2 + \dots + X_n$$

i.e. $S_n(\omega) = \sum_{j=1}^n \omega_j$

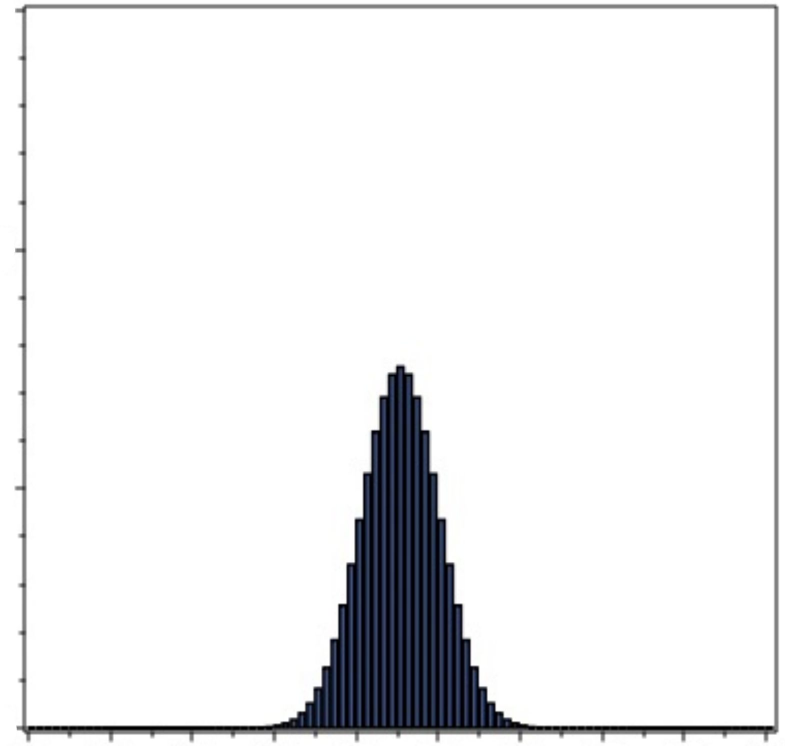
$$\mathbb{P}(S_n = k) = \frac{1}{2^n} \#\{\omega : \omega_1 + \dots + \omega_n = k\}$$

What about the proportion of Heads?

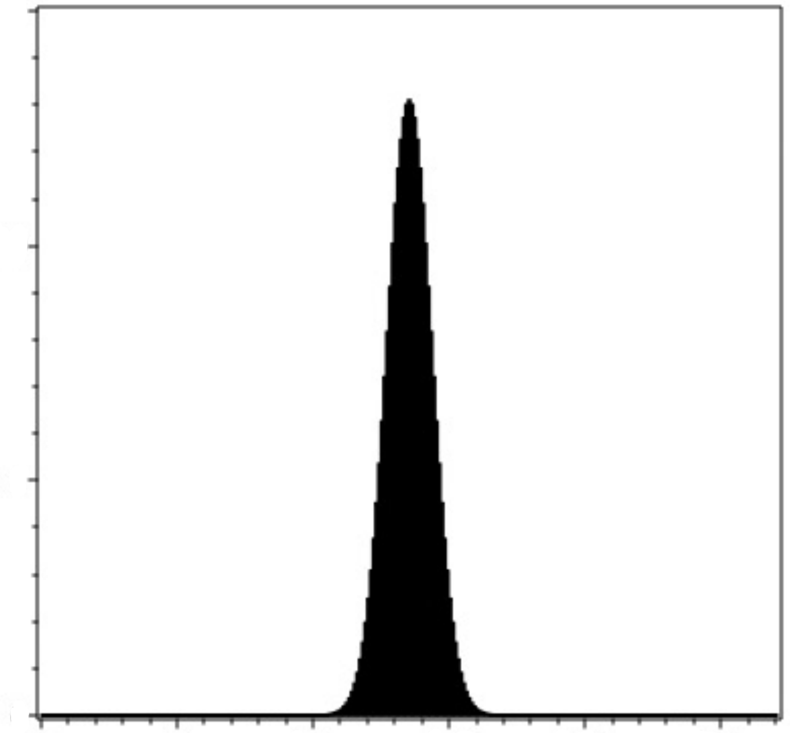
$$\frac{S_n}{n}$$



$\frac{S_{30}}{30}$



$\frac{S_{90}}{90}$



$\frac{S_{270}}{270}$

This concentration isn't really specific to $\mathrm{Binom}(\frac{1}{2}, n)$ distributions. It's because of the normalization S_n / n and, crucially,

$$S_n = X_1 + X_2 + \cdots + X_n$$

The Weak Law of Large Numbers

Theorem. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of L^2 random variables on a probability space, that are pairwise uncorrelated:

$$\mathrm{Cov}(X_n, X_m) = 0 \quad \text{if } n \neq m$$

and all with the same mean and variance:

$$\mathbb{E}[X_n] = \alpha, \quad \mathrm{Var}[X_n] = t \quad \forall n.$$

Let $S_n = X_1 + \cdots + X_n$. Then for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left| \frac{S_n}{n} - \alpha \right| \geq \varepsilon \right)$$

Pf.

What kind of limit is this?

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n}-\alpha\right|\geq\varepsilon\right)=0, \quad \forall\, \varepsilon>0.$$

It means $\frac{S_n}{n}$ is asymptotically concentrated at α.

But does it mean $\frac{S_n}{n}\to\alpha$ a.s.?

Eg. Suppose $X_1, X_2, X_3, \dots$ are coin tosses, with $\mathbb{P}(X_n=1)=\frac{1}{n}$

$$\mathbb{P}(|X_n|\geq\varepsilon)$$

Does $X_n\to 0$ a.s.?