L^2 [§10.5 in Driver]

Given a measure space $(\Omega, \mathcal{F}, \mu)$,

$L^2(\Omega, \mathcal{F}, \mu) = \{ f : \Omega \to \mathbb{R}$ measurable s.t. $\int_\Omega f^2 \, d\mu < \infty \}$.

Note: for any real numbers f, g,

$$(|f| - |g|)^2 =$$

Thus, if $f, g \in L^2$, then

I.e. **Prop**: If $f, g \in L^2$ then

Cor: If μ is a finite measure, then $L^2(\mu) \subseteq L^1(\mu)$.

Cauchy-Schwarz

For $f, g \in L^2(\Omega, \mathcal{F}, \mu)$, define:

$$\|f\|_{L^2} := \left(\int_\Omega f^2 \, d\mu\right)^{1/2}, \qquad \langle f, g \rangle_{L^2} := \int_\Omega fg \, d\mu$$

Theorem: If $f, g \in L^2(\Omega, \mathcal{F}, \mu)$, then

$$|\langle f, g \rangle_{L^2}| \leq \int |fg| \, d\mu \leq \|f\|_{L^2} \|g\|_{L^2}.$$

Pf.

Cor: $L^2(\Omega, \mathcal{F}, \mu)$ is a vector space, and $\|f\|_{L^2} = \sqrt{\langle f, f \rangle_{L^2}}$ is a norm on it.

Pf.

Fact: L^2 is actually a Hilbert space: it is (Cauchy) complete.

I.e. If $f_n \in L^2$ s.t. $\|f_n - f_m\|_{L^2} \to 0$ as $n,m \to \infty$,
then $\exists!\, f \in L^2$ s.t. $\|f_n - f\|_{L^2} \to 0$.

(The same holds true in L^1, and in general in L^p for $1 \le p \le \infty$.)

We'll return to prove this important fact in the near future.

This will be very important when we make a serious study of conditional probability.

Covariance

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $L^2 \subseteq L^1$. In fact

$$\|X\|_{L^1} = \int_\Omega |X| \, d\mathbb{P}$$

Def For $X, Y \in L^2$, let $\mathring{X} = X - \mathbb{E}[X]$, $\mathring{Y} = Y - \mathbb{E}[Y]$

Their **covariance** is

$$\mathrm{Cov}(X, Y) := \mathbb{E}[\mathring{X}\mathring{Y}]$$

For $X \in L^2$, its **variance** is

$$\mathrm{Var}(X) := \mathrm{Cov}(X, X) = \mathbb{E}[\mathring{X}^2]$$

Lemma: If $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $Var(X) = 0$,
then $X \equiv$ Const. a.s.

Pf.

Eg. $X \overset{d}{=} \mathcal{N}(\alpha, t)$. $\quad X \overset{d}{=} \sqrt{t} Z + \alpha$, $\quad Z \overset{d}{=} \mathcal{N}(0,1)$

$$Var(X) =$$

Eg. $T \overset{d}{=} \text{Exp}(\alpha)$, $\quad \mathbb{E}[T] = \frac{1}{\alpha}$

Eg. $N \overset{d}{=} \text{Pois}(\alpha)$, $\quad \mathbb{E}[N] = \alpha$

Def: $X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ are uncorrelated if $\mathrm{Cov}(X,Y) = 0$.

In general, their correlation is $\mathrm{Corr}(X,Y) = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$

Prop: $\mathrm{Cov}(X+\alpha, Y) = \mathrm{Cov}(X, Y+\alpha) = \mathrm{Cov}(X,Y)$ for any $\alpha \in \mathbb{R}$. As a result, if $X_1, \dots, X_n$ are all (pairwise) uncorrelated, then

$$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n).$$

Pf.

What Does "Uncorrelated" Mean?

Eg. Let $A, B \in \mathcal{F}$, and consider $\mathbb{1}_A, \mathbb{1}_B$

$$\operatorname{Cov}(\mathbb{1}_A, \mathbb{1}_B)$$
$$= \mathbb{E}(\mathbb{1}_A \mathbb{1}_B) - \mathbb{E}(\mathbb{1}_A)\mathbb{E}(\mathbb{1}_B)$$

Eg. Toss a fair coin n times. $X_j = \begin{cases} 1 & \text{if the } j^{th} \text{ toss is Heads} \\ 0 & \text{" " Tails} \end{cases}$

$$\Omega = \{(\omega_1, \ldots, \omega_n) : \omega_j \in \{0, 1\}\}$$
$$\mathcal{F} = 2^{\Omega}$$
$$\mathbb{P}(A) = \frac{\# A}{2^n}$$