Expectation - Back to the Beginning

Ω finite, $X : \Omega \to \mathbb{R}$, $\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\{\omega\})$

If we also have $g : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g(X(\omega))\, \mathbb{P}(\{\omega\})$

Change of Variables

Proposition: Let $X : (\Omega, \mathcal{F}, \mu) \to (S, \mathcal{B})$, $\nu = X_* \mu$

Then $g : S \to \mathbb{R}$ is in $L^1(S, \mathcal{B}, \nu)$
iff $g \circ X : \Omega \to \mathbb{R}$ is in $L^1(\Omega, \mathcal{F}, \mu)$, and

$$\int_S g \, d\nu = \int_\Omega (g \circ X) \, d\mu$$

Pf. [HW6]

In particular, take $(S, \mathcal{B}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $\underline{X} : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^d$
a Borel random vector. Then

$\underline{X}_* \mathbb{P}$

$$\int g \circ \underline{X} \, d\mathbb{P} = \int_{\mathbb{R}^d} g \, d(\underline{X}_* \mathbb{P})$$

In particular, for $\mathbb{R}$-valued random variables,

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g \, d\mu_X .$$

If $F_X(t) = \mathbb{P}(X \leq t)$ is the CDF, then $\mu_X = \mu_{F_X}$ and so if g happens to be Riemann-Stieltjes integrable (e.g. continuous),

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(t) \, dF_X(t)$$

Upshot: if we know μ_X, we can calculate expectations of functions of X.

Eg. (moments) $\mathbb{E}[X^n] =$

E.g. $\mathcal{N}(\alpha, t)$ Normal distributions.

Density $\frac{1}{\sqrt{2\pi t}} e^{-(x-\alpha)^2/2t}$

Fact: If $X \overset{d}{=} \mathcal{N}(\alpha, t)$ and $Z \overset{d}{=} \mathcal{N}(0,1)$, then $X \overset{d}{=} \sqrt{t}\, Z + \alpha$.

Let $Z \overset{d}{=} \mathcal{N}(0,1)$. Then $\mathbb{E}[Z] = \int_{\mathbb{R}} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

$$\mathbb{E}[Z^2] = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

For higher moments, we can use Gaussian integration by parts:

$$\mathbb{E}[Z f(Z)] = \mathbb{E}[f'(Z)]$$

$$\therefore \mathbb{E}[Z^{n+2}] = \mathbb{E}[Z \cdot Z^{n+1}] = (n+1)\mathbb{E}[Z^n]$$

$$\hookrightarrow \mathbb{E}[Z^n] = \begin{cases} \end{cases}$$

E.g. Exponential distributions $\mathrm{Exp}(\alpha)$, $\alpha > 0$

$T \overset{d}{=} \mathrm{Exp}(\alpha)$ if $F_T(t) = (1 - e^{-\alpha t}) \mathbb{1}_{t \geq 0}$

Often models a "waiting time"

$\mathbb{P}(T > t) =$

$\mathbb{E}[T] = \int_0^\infty t \cdot \alpha e^{-\alpha t} \, dt$

Eg. Poisson distribution $\mathrm{Pois}(\alpha)$

$N \overset{d}{=} \mathrm{Pois}(\alpha)$ iff $N : \Omega \to \mathbb{N}$ and

$$\mathbb{P}(N=k) = e^{-\alpha} \frac{\alpha^k}{k!}$$

Models number of events occurring in a fixed time period.

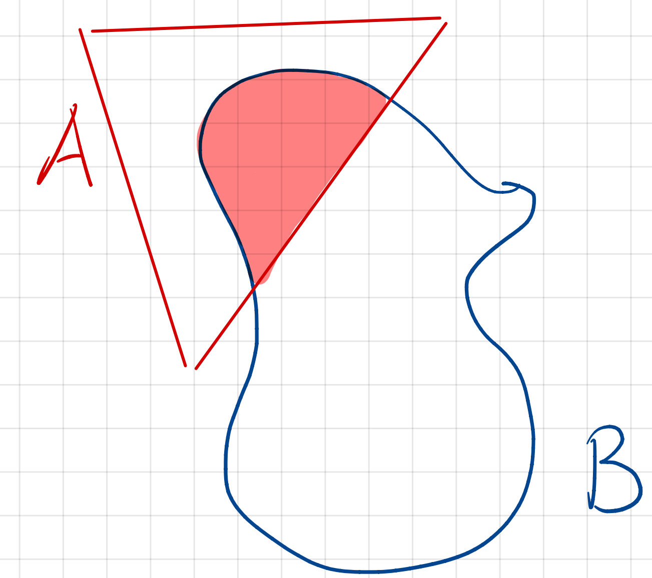
Closely related to $\mathrm{Exp}(\alpha)$; we'll look into this later.

$$\mathbb{E}[N] = \int_{\mathbb{N}} t \cdot \mu_N(dt)$$

Eg. Uniform Random Variables / Vectors

Let $B \in \mathcal{B}(\mathbb{R}^d)$ with $\lambda(B) > 0$.

A random vector $X \overset{d}{=} \mathrm{Unif}(B)$ iff $\mathbb{P}(X \in A) = \dfrac{\lambda(A \cap B)}{\lambda(B)}$



For example: $U \overset{d}{=} \mathrm{Unif}([a,b])$

$\mathbb{E}[U] =$