When is there a Density?

Let μ, ν be two measures on $(\Omega, \mathcal{F})$.

How can we tell if $\exists$ density $\rho \in L^+(\Omega, \mathcal{F})$ s.t.

$$\text{"} d\nu = \rho \, d\mu \text{"}$$

$$\nu(A) = \int_A \rho \, d\mu \quad \forall A \in \mathcal{F}.$$

There is one fairly straightforward necessary condition:

Suppose $A \in \mathcal{F}$ and $\mu(A) = 0$

$$\nu(A) = \int_A \rho \, d\mu = 0.$$

$$\text{"} 0 \cdot \infty = 0 \text{"}$$

Def: Say $\nu \ll \mu$, i.e. ν is absolutely continuous wrt μ if $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Theorem (Radon-Nikodym)

Let μ, ν be σ-finite measures on $(\Omega, \mathcal{F})$.
Then $\nu \ll \mu$ if and only if $\exists\, \rho : \Omega \to [0, \infty)$ measurable s.t.

$$\nu(A) = \int_A \rho \, d\mu \quad \forall A \in \mathcal{F}$$

I.e. $d\nu = \rho \, d\mu$

Moreover, this density ρ is uniquely defined up to a μ-null set. It is called the
Radon-Nikodym derivative

not a good proof!

$$\rho = \frac{d\nu}{d\mu}$$

$\leftarrow$ If $(\Omega, \mathcal{F}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and μ, ν are Radon-measures

$$\frac{d\nu}{d\mu}(x) = \lim_{\varepsilon \downarrow 0} \frac{\nu(B_\varepsilon(x))}{\mu(B_\varepsilon(x))} \quad \mu\text{-a.e. } x \in \mathbb{R}^d.$$

Theorem (Lebesgue) [20.8]

Let μ, ν be σ-finite measures on $(\Omega, \mathcal{F})$.
Then ν has a unique Lebesgue decomposition

$$\nu = \nu_a + \nu_s$$

where

- $\nu_a \ll \mu$ (and $\therefore$ $\exists\, \rho \in L^+(\Omega, \mathcal{F})$ s.t. $d\nu_a = \rho\, d\mu$)
- $\nu_s \perp \mu$: ν_s and μ are mutually singular, meaning $\exists\, A \in \mathcal{F}$ s.t. $\nu_s(A) = 0$ and $\mu(A^c) = 0$.

Eg. $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

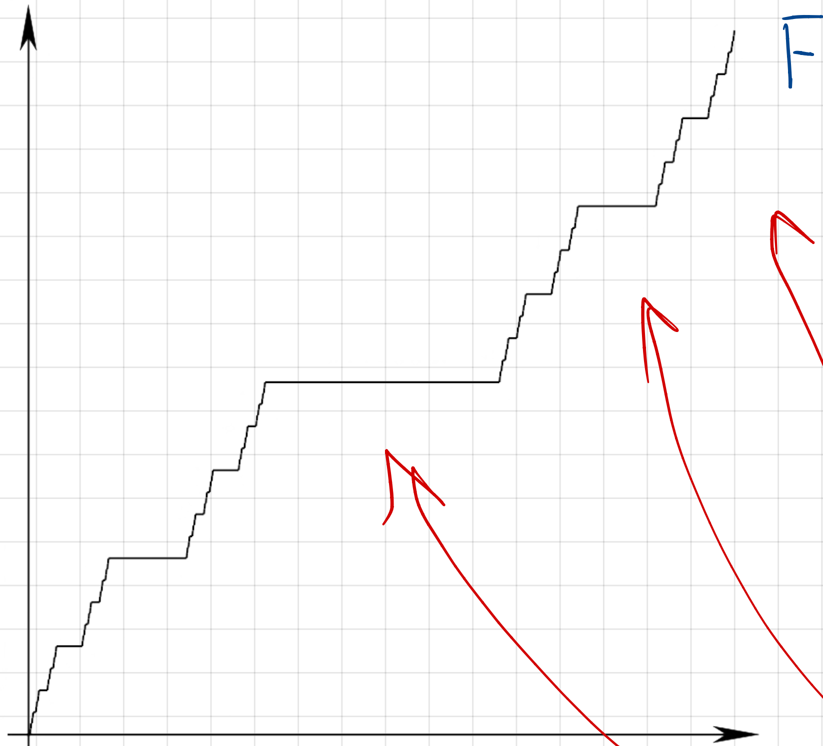
$$\nu(A) = \int_A \rho\, d\lambda + \overbrace{\underbrace{\sum_{j=1}^{\infty} \alpha_j \delta_{x_j}(A)}_{\perp \lambda}}^{\nu_s}, \quad x_j \in \mathbb{R} \quad \alpha_j \geq 0$$

$d\nu_a = \rho\, d\lambda$

$\lambda(\{x_j\}_{j=1}^{\infty}) = 0, \quad \nu_s(\mathbb{R} \setminus \{x_j\}_{j=1}^{\infty}) = 0.$

Not every singular measure is discrete!

E.g. The Devil's Staircase



The Radon measure μ_F has no point masses – F is continuous. But still: $\mu_F \perp \lambda$:

$\mu_F(C^c) = 0$ but $\lambda(C) = 0$.

More Precise Lebesgue Decomposition:

$$\nu = \nu_a + \nu_{pp} + \nu_{sc} \leftarrow \text{singular continuous}$$

$\nu_a \ll \mu$

"pure point" $\sum_{j=1}^{\infty} \alpha_j \delta_{\omega_j}$

$\nu_{sc} \perp \mu, \quad \nu_{sc}(\{\omega\}) = 0 \quad \forall \omega \in \Omega$.