Lebesgue vs. Riemann

Let $(\mathbb{R}, \mathcal{B}, \lambda)$ be the Lebesgue measure on Borel sets in $\mathbb{R}$. We now know how to define $\int_{\mathbb{R}} f \, d\lambda$ for $f \in L^1(\Omega, \mathcal{B}, \mathbb{R})$.

Eg. $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$

So we can integrate non-Riemann integrable things.

Thm: [11.5] Let $\overline{\mathcal{B}}$ denote the completion of $\mathcal{B}(\mathbb{R})$ wrt λ. Then a bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable iff it is $\overline{\mathcal{B}}/\mathcal{B}$ measurable, and $\lambda\{x \in [a,b] : f \text{ is discontinuous @ } x\} = 0$.

In this case, $\int_{[a,b]} f \, d\lambda = \int_a^b f(x)\,dx$.

Partial Proof:

The Lebesgue integral also allows us to handle "improper integrals".

E.g. $\int_{-\infty}^{\infty} f(x)\,dx := \lim_{n\to\infty} \int_{-n}^{n} f(x)\,dx$

What about other Borel measures on $\mathbb{R}$?

So long as $\mu[-n,n] < \infty$ $\forall n \in \mathbb{N}$, we mean Radon measures, and so $\mu = \mu_F$ i.e. $\mu(a,b] = F(b) - F(a)$.

F is right-continuous. Suppose that F is

Of course, not every Radon measure has a density.

E.g. $\mu = \delta_x = \mu_F$ with $F = \mathbb{1}_{(x,\infty)}$.

We see that F had better be continuous (i.e. μ_F has no point mass) if we want μ_F to possess a density.

To mimic the calculations on the last page, we may not need $F \in C^1$, but we at least need the Fundamental Theorem of Calculus to hold

In general: if F is continuous, differentiable a.e., and nice enough that

$$F(x) = \int_{[a,x]} F' \, d\lambda \quad \text{for } \lambda\text{-a.e. } x$$

then we can mimic the preceding to see that

$$\mu_F(A) = \int_A F' \, d\lambda$$

If $F \in C^1$, this is fine. It works much more generally - but it doesn't always work, even if F is continuous, and diff'ble a.e.

Eg. The Devil's Staircase F diff'ble a.e., $F' = 0$.