

Lebesgue vs. Riemann

Let $(\mathbb{R}, \mathcal{B}, \lambda)$ be the Lebesgue measure on Borel sets in \mathbb{R} . We now know how to define $\int_{\mathbb{R}} f d\lambda$ for $f \in L^1(\Omega, \overline{\mathcal{B}}, \mathbb{R})$.

Eg. $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$ $\leftarrow \mathbb{Q} \cap [0,1] \in \mathcal{B}(\mathbb{R})$

$\therefore \in L^1$

$$\int f d\lambda = \int \mathbb{1}_{\mathbb{Q} \cap [0,1]} d\lambda = \lambda(\mathbb{Q} \cap [0,1]) = 0.$$

\uparrow
countable

So we can integrate non-Riemann integrable things.

Thm: [11.5] Let $\overline{\mathcal{B}}$ denote the completion of $\mathcal{B}(\mathbb{R})$ wrt λ . Then a bounded function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable iff it is $\overline{\mathcal{B}}/\mathcal{B}$ measurable, and $\lambda\{x \in [a, b] : f \text{ is discontinuous @ } x\} = 0$.

In this case,
$$\int_{[a, b]} f d\lambda = \int_a^b f(x) dx$$

Partial Proof: Suppose f is Riemann int'ble. (t_{j-1}, t_j] = I_j \quad |I_j| = t_j - t_{j-1} = \lambda(I_j)

$\therefore \forall \varepsilon > 0$ can find $\pi = \{a = t_0 < t_1 < \dots < t_n = b\}$

$$\int f d\lambda - \varepsilon \leq \int_{\pi} f - \varepsilon \leq \int_a^b f(x) dx \leq \int_{\pi} f + \varepsilon \leq \int f d\lambda + \varepsilon$$

$$\overline{\varphi}_{\pi} = \sum_{j=1}^n \sup_{I_j} f \cdot \mathbb{1}_{I_j} \geq f \quad \sum_{j=1}^n \inf_{I_j} f \cdot |I_j| = \int \underline{\varphi}_{\pi} d\lambda$$

$$\int f d\lambda \leq \int \overline{\varphi}_{\pi} d\lambda$$

$$\underline{\varphi}_{\pi} = \sum_{j=1}^n \inf_{I_j} f \cdot \mathbb{1}_{I_j} \leq f$$

The Lebesgue integral also allows us to handle "improper integrals".

$$\text{E.g. } \int_{-\infty}^{\infty} f(x) dx := \lim_{n \rightarrow \infty} \int_{-n}^n f(x) dx$$

$$\uparrow \\ \infty \\ \text{iff} \\ f \in L^1(\lambda) \\ = \lim_{n \rightarrow \infty} \int_{[-n, n]} f d\lambda$$

$$= \lim_{n \rightarrow \infty} \int f \mathbb{1}_{[-n, n]} d\lambda$$

$$= \int \lim_{n \rightarrow \infty} f \mathbb{1}_{[-n, n]} d\lambda$$

$$= \int f d\lambda,$$

Assume $f \geq 0$ continuous

$$\mathbb{1}_{[-n, n]} \leq \mathbb{1}_{[-n-1, n+1]}$$

$$\therefore f \mathbb{1}_{[-n, n]} \leq f \mathbb{1}_{[-n-1, n+1]}$$

↙ MCT

What about other Borel measures on \mathbb{R} ?

So long as $\mu[-n, n] < \infty \quad \forall n \in \mathbb{N}$, we mean

↙ FT ↗ $\therefore F' \geq 0$.

Radon measures, and so $\mu = \mu_F$ i.e. $\mu(a, b] = F(b) - F(a)$.

F is right-continuous. Suppose that F is $C^1(\mathbb{R})$

$$\begin{aligned}\mu(a, b] = F(b) - F(a) &= \int_a^b F'(x) dx \\ &= \int_{[a, b]} F' d\lambda.\end{aligned}$$

Define $\nu(A) = \int_A F' d\lambda$. Then $\nu = \mu$ on $\mathcal{C}(I)$.

$\therefore \nu = \mu$ on $\sigma(\mathcal{C}(I)) = \mathcal{B}(\mathbb{R})$.

$$\begin{aligned}\therefore \int_{[a, b]} f d\mu &= \int_{[a, b]} f F' d\lambda \\ &= \int_a^b f(x) F'(x) dx.\end{aligned} \quad \forall f \in L^1([a, b], \mu) \Leftrightarrow \begin{array}{l} \text{if } |F'| \in L^1([a, b], \lambda) \\ \uparrow \\ \text{continuous.} \end{array}$$

Of course, not every Radon measure has a density.

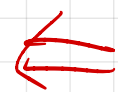
E.g. $\mu = \delta_x = \mu_F$ with $F = \mathbb{1}_{(x, \infty)}$.

$$\mu(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} = \int_A \rho \, d\lambda$$

if $(a, b] \not\ni x$ $\rho \geq 0$.

$\therefore \int_{(a, b]} \rho \, d\lambda = 0 \Rightarrow \rho = 0$ a.e. $[x]$ on $(a, b]$.

$$\int_A \rho \, d\lambda = 0.$$



$\therefore \rho = 0$ a.e. $[x]$ on $\mathbb{R} \setminus \{x\}$
 $\Leftrightarrow \rho = 0$ a.e. $[x]$ on \mathbb{R}

We see that F had better be continuous (i.e. μ_F has no point mass) if we want μ_F to possess a density.

To mimic the calculations on the last page, we may not need $F \in C^1$, but we at least need the Fundamental Theorem of Calculus to hold

In general: if F is continuous, differentiable a.e., and nice enough that

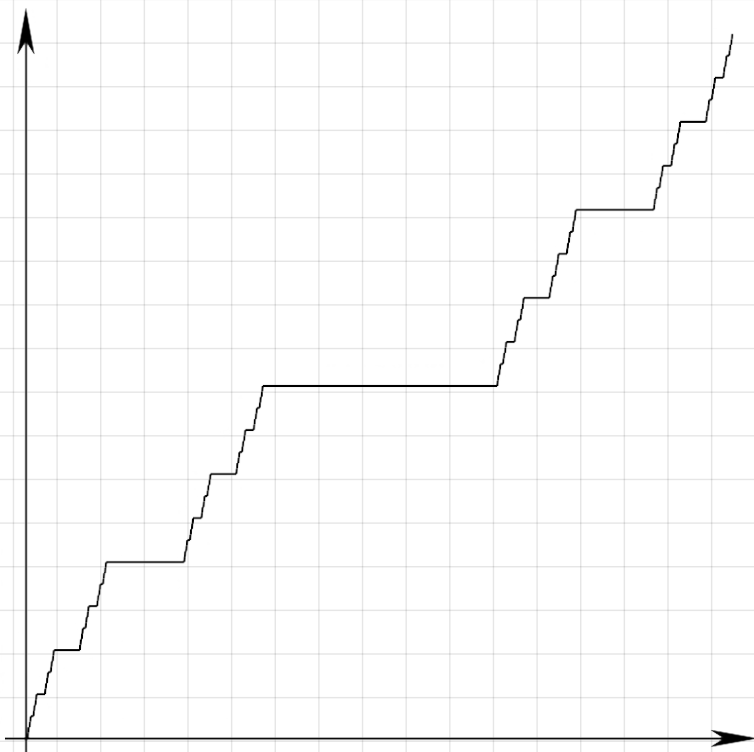
$$F(x) = \int_{[0,x]} F' d\lambda \quad \text{for } \lambda\text{-a.e. } x$$

then we can minimize the preceding to see that

$$\mu_F(A) = \int_A F' d\lambda$$

If $F \in C^1$, this is fine. It works much more generally - but it doesn't always work, even if F is continuous, and diff'ble a.e.

Eq. **The Devil's Staircase** F diff'ble a.e., $F' = 0$.



$$\text{But } \int_{[0,x]} F' d\lambda = 0 = F(x) \quad \forall x > 0.$$