Lebesgue vs. Riemann

Let $(\mathbb{R}, \mathcal{B}, \lambda)$ be the Lebesgue measure on Borel sets in $\mathbb{R}$. We now know how to define $\int_{\mathbb{R}} f \, d\lambda$ for $f \in L^1(\Omega, \overline{\mathcal{B}}, \mathbb{R})$.

E.g. $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$ $\leftarrow$ $\mathbb{Q} \cap [0,1] \in \mathcal{B}(\mathbb{R})$

$\therefore \quad \in L^1$

$$\int f \, d\lambda = \int \mathbb{1}_{\mathbb{Q} \cap [0,1]} \, d\lambda = \lambda(\mathbb{Q} \cap [0,1]) = 0.$$

countable

So we can integrate non-Riemann integrable things.

Thm [11.5] Let $\overline{\mathcal{B}}$ denote the completion of $\mathcal{B}(\mathbb{R})$ wrt λ. Then a bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable iff it is $\overline{\mathcal{B}}/\mathcal{B}$ measurable, and

$$\lambda\{x \in [a,b] : f \text{ is discontinuous @ } x\} = 0.$$

In this case, $\displaystyle\int_{[a,b]} f \, d\lambda = \int_a^b f(x)\,dx$.

$(t_{j-1}, t_j] = I_j \quad |I_j| = t_j - t_{j-1}$
$= \lambda(I_j)$ ✓

Partial Proof: S'pose f is Riemann int'ble.

$\therefore \forall \varepsilon > 0$ can find $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$

$$\int f d\lambda - \varepsilon \;\le\; \overline{\int_\pi} f - \varepsilon \;<\; \int_a^b f(x)\,dx \;\le\; \underline{\int_\pi} f + \varepsilon \;\therefore\; \le \int f d\lambda + \varepsilon$$

$$\overline{\varphi}_\pi = \sum_{j=1}^n \sup_{I_j} f \, \mathbb{1}_{I_j} \ge f \qquad \sum_{j=1}^n \inf_{I_j} f \cdot |I_j| = \int \underline{\varphi}_\pi \, d\lambda$$

$$\int f d\lambda \le \int \overline{\varphi}_\pi \, d\lambda \qquad \underline{\varphi}_\pi = \sum_{j=1}^n \inf_{I_j} f \, \mathbb{1}_{I_j} \le f$$

The Lebesgue integral also allows us to handle "improper integrals".

Eg. $\int_{-\infty}^{\infty} f(x)\,dx := \lim_{n\to\infty} \int_{-n}^{n} f(x)\,dx$

$$= \lim_{n\to\infty} \int_{[-n,n]} f \, d\lambda$$

$$= \lim_{n\to\infty} \int f \mathbb{1}_{[-n,n]} \, d\lambda$$

$$= \int \lim_{n\to\infty} f \mathbb{1}_{[-n,n]} \, d\lambda$$

$$= \int f \, d\lambda,$$

Λ

∞

iff

$f \in L^1(\lambda)$

Assume $f \geq 0$ continuous

$$\mathbb{1}_{[-n,n]} \leq \mathbb{1}_{[-n-1,n+1]}$$

$$\therefore f \mathbb{1}_{[-n,n]} \leq f \mathbb{1}_{[-n-1,n+1]}$$

$\swarrow$ MCT

What about other Borel measures on $\mathbb{R}$?

So long as $\mu[-n,n] < \infty$ $\forall n \in \mathbb{N}$, we mean Radon measures, and so $\mu = \mu_F$ i.e. $\mu(a,b] = F(b) - F(a)$. $F\uparrow \therefore F' \geq 0$.

F is right-continuous. Suppose that F is $C^1(\mathbb{R})$

$$\mu(a,b] = F(b) - F(a) = \int_a^b F'(x)\,dx.$$

$$= \int_{[a,b]} F'\,d\lambda.$$

Define $\nu(A) = \int_A F'\,d\lambda$. Then $\nu = \mu$ on $\mathcal{C}_{(]}$.

$\therefore$ $\nu = \mu$ on $\sigma(\mathcal{C}_{(]}) = \mathcal{B}(\mathbb{R})$.

$$\therefore \int_{[a,b]} f\,d\mu = \int_{[a,b]} f F'\,d\lambda \qquad \forall f \in L^1([a,b],\mu) \Leftrightarrow |f|F' \in L^1([a,b],\lambda)$$

F' continuous.

$$= \int_a^b f(x)\,F'(x)\,dx.$$

Of course, not every Radon measure has a density.

E.g. $\mu = \delta_x = \mu_F$ with $F = \mathbb{1}_{(x,\infty)}$.

$$\mu(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} = \int_A \rho \, d\lambda$$

if $(a,b] \not\ni x$ $\quad \rho \geq 0$.

$\therefore \int_{(a,b]} \rho \, d\lambda = 0 \Rightarrow \rho = 0$ a.e.$[\lambda]$ on $(a,b]$.

$\therefore \rho = 0$ a.e. $[\lambda]$ on $\mathbb{R} \setminus \{x\}$

$\int_A \rho \, d\lambda = 0 \quad \Leftarrow \quad \Leftrightarrow \rho = 0$ a.e. $[\lambda]$ on $\mathbb{R}$

We see that F had better be continuous (i.e. μ_F has no point mass) if we want μ_F to possess a density.

To mimic the calculations on the last page, we may not need $F \in C^1$, but we at least need the Fundamental Theorem of Calculus to hold

In general: if F is continuous, differentiable a.e., and nice enough that

$$F(x) = \int_{[a,x]} F' \, d\lambda \quad \text{for } \lambda\text{-a.e. } x$$

then we can mimic the preceding to see that

$$\mu_F(A) = \int_A F' \, d\lambda$$

If $F \in C^1$, this is fine. It works much more generally - but it doesn't always work, even if F is continuous, and diff'ble a.e.

Eg. The Devil's Staircase F diff'ble a.e., $F' = 0$.

But $\int_{[0,x]} F' \, d\lambda = 0 \neq F(x) \quad \forall x > 0$.