1-Parameter Families of Random Variables

We will often have occasion to consider 1-parameter families of random variables

$$\{X_t : t \in (a,b)\} \qquad -\infty \leq a < b \leq \infty$$

We will study many stochastic processes of this form, but they also come up more innocuously.

Eg. Let X be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. Then $e^{tX} \geq 0$ for all $t \in \mathbb{R}$, so

$$\mathbb{E}[e^{tX}] = \int e^{tX} \, d\mathbb{P} \quad \text{exists in } [0,\infty]$$

Note: if $|t| < \varepsilon$ then $tX \leq |tX| \leq \varepsilon |X|$

Prop. [10.31] Let $(\Omega, \mathcal{F}, \mu)$ be a measure space,

and $f : (a,b) \times \Omega \to \mathbb{R}$ s.t.

1. $\omega \mapsto f(t,\omega)$ is measurable for each $t \in (a,b)$
2. $f(t_0, \cdot) \in L^1(\Omega, \mathcal{F}, \mu)$ for some $t_0 \in (a,b)$
3. $\partial f / \partial t\,(t,\omega)$ exists for μ-a.e. ω and for every $t \in (a,b)$
4. There is $g \in L^1(\Omega, \mathcal{F}, \mu)$ s.t. $\left|\frac{\partial f}{\partial t}(t,\omega)\right| \leq g(\omega)$ for μ-a.e. ω and for every $t \in (a,b)$.

Then $f(t,\cdot) \in L^1$ for all $t \in (a,b)$,

$t \mapsto \int f(t,\omega)\,\mu(d\omega)$ is differentiable on (a,b),

and
$$\frac{d}{dt} \int f(t,\omega)\,\mu(d\omega) = \int \frac{\partial f(t,\omega)}{\partial t}\,\mu(d\omega).$$

Pf. $\frac{\partial f}{\partial t}(t,\omega) = \lim_{n\to\infty} n \cdot \left[f(t+\frac{1}{n},\omega) - f(t,\omega) \right]$

Mean Value Thm: $\left| \frac{f(t,\omega) - f(t_0,\omega)}{t - t_0} \right|$

$\therefore \ |f(t,\omega)| = | f(t,\omega) - f(t_0,\omega) + f(t_0,\omega) |$

Define $G(t) = \int f(t,\omega)\, \mu(d\omega)$. Then

$$\frac{G(t) - G(t_0)}{t - t_0}$$

E.g. Let $f(t,\omega) = e^{tX(\omega)}$.

If $e^{\varepsilon|X|} \in L^1$, then so is $f(t_0,\cdot)$ for (say) $t_0 = \frac{\varepsilon}{2}$

$$\frac{\partial f}{\partial t} = \frac{d}{dt} e^{tX} = Xe^{tX}$$

Actually, we can do this all at once. Note: $e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n$.

Prop: If $e^{\varepsilon|X|} \in L^1$ then M_X is analytic on $(-\varepsilon, \varepsilon)$ and

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n] \qquad |t| < \varepsilon.$$

Pf.

Caution:

It's really important that the conditions of the differentiate-under-an-integral proposition held a.s. independently of t.

E.g. Lebesgue measure $([0,1], \mathcal{B}([0,1]), \lambda)$

$$f(t,\omega) = \mathbb{1}_{\omega \le t}$$

1. For fixed t,

2. $|f(t,\cdot)| \le 1$ and $\lambda([0,1]) = 1 < \infty$ so $f(t,\cdot) \in L^1$

3. $\frac{\partial f}{\partial t} =$

4.

$$\frac{d}{dt}\int \mathbb{1}_{[0,t]}\, d\lambda = \frac{d}{dt}\int f(t,\omega)\,\lambda(d\omega) = \int \frac{\partial f}{\partial t}\, d\lambda$$