1-Parameter Families of Random Variables

We will often have occasion to consider 1-parameter families of random variables

$$\{X_t : t \in (a,b)\} \qquad -\infty \le a < b \le \infty$$

We will study many stochastic processes of this form, but they also come up more innocuously.

Eg. Let X be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$.

Then $e^{tX} \ge 0$ for all $t \in \mathbb{R}$, so

$M_X(t) = \mathbb{E}[e^{tX}] = \int e^{tX}\, d\mathbb{P}$ exists in $[0,\infty]$

Note: if $|t| < \varepsilon$ then $tX \le |tX| \le \varepsilon |X|$

If $\exists\, \varepsilon > 0$ s.t. $e^{\varepsilon|X|} \in L^1$ $\therefore \int e^{tX}\, d\mathbb{P} \le \int e^{\varepsilon|X|}\, d\mathbb{P} = \mathbb{E}[e^{\varepsilon|X|}] < \infty$.

$\therefore e^{tX} \in L^1 \quad \forall t \in (-\varepsilon, \varepsilon)$.

Prop [10.31] Let $(\Omega, \mathcal{F}, \mu)$ be a measure space,

and $f: (a,b) \times \Omega \to \mathbb{R}$ s.t.

1. $\omega \mapsto f(t,\omega)$ is measurable for each $t \in (a,b)$
2. $f(t_0, \cdot) \in L^1(\Omega, \mathcal{F}, \mu)$ for some $t_0 \in (a,b)$
3. $\partial f / \partial t \, (t,\omega)$ exists for μ-a.e. ω and for every $t \in (a,b)$
4. There is $g \in L^1(\Omega, \mathcal{F}, \mu)$ s.t. $\left| \frac{\partial f}{\partial t}(t,\omega) \right| \leq g(\omega)$ for μ-a.e. ω and for every $t \in (a,b)$.

Then $f(t, \cdot) \in L^1$ for all $t \in (a,b)$,

$t \mapsto \int f(t,\omega) \mu(d\omega)$ is differentiable on (a,b),

and

$$\frac{d}{dt} \int f(t,\omega) \mu(d\omega) = \int \frac{\partial f(t,\omega)}{\partial t} \mu(d\omega) .$$

Pf. $\frac{\partial f}{\partial t}(t,\omega) = \lim_{n\to\infty} n\cdot\left[f(t+\frac{1}{n},\omega) - f(t,\omega)\right]$

between t, t_0

Mean Value Thm: $\left|\frac{f(t,\omega)-f(t_0,\omega)}{t-t_0}\right| = \left|\frac{\partial f}{\partial t}(t^*,\omega)\right| \le g(\omega)$

$\therefore |f(t,\omega)| = |f(t,\omega) - f(t_0,\omega) + f(t_0,\omega)|$

$\le |f(t,\omega)-f(t_0,\omega)| + |f(t_0,\omega)| \le \underbrace{|t-t_0|\,g(\omega)}_{L^1} + \underbrace{|f(t_0,\omega)|}_{L^1}$ $\therefore \in L^1$.

Define $G(t) = \int f(t,\omega)\,\mu(d\omega)$. Then

$$\frac{G(t)-G(t_0)}{t-t_0} = \int \frac{f(t,\omega)-f(t_0,\omega)}{t-t_0}\,\mu(d\omega)$$

$$\lim_{t\to t_0} \frac{G(t)-G(t_0)}{t-t_0} = \lim_{t\to t_0}\int \text{''} \qquad \text{''}$$

$$\overset{\text{WTS}}{=} \int \lim_{t\to t_0}\Big(\qquad\Big)\,\mu(d\omega)$$

by DCT.

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E.g. Let $f(t,\omega) = e^{tX(\omega)}$. measurable $\forall t$ ✓

If $e^{\varepsilon|X|} \in L^1$, then so is $f(t_0,\cdot)$ for (say) $t_0 = \frac{\varepsilon}{2}$

$\frac{\partial f}{\partial t} = \frac{d}{dt} e^{tX} = X e^{tX}$ exists $\forall t$ ✓

Let $\alpha \in (0,1)$ Then $|X| \leq e^{\alpha\varepsilon|X|}$

$\therefore |X e^{tX}| \leq e^{\alpha\varepsilon|X|} \cdot e^{(1-\alpha)\varepsilon|X|} = e^{\varepsilon|X|} \in L^1$

Let $|t| < (1-\alpha)\varepsilon$

$\therefore \left|\frac{\partial f}{\partial t}\right| \leq e^{\varepsilon|X|}$ $\forall t$ $|t| < (1-\alpha)\varepsilon$ $\forall \alpha < 1$

$\therefore \alpha \uparrow 1$ $|t| < \varepsilon$.

$\therefore$ by [10.3.1] $\frac{d}{dt} M_X(t) = \frac{d}{dt} E[e^{tX}] = E[\frac{\partial}{\partial t} e^{tX}] = E[X e^{tX}]$

Repeat: $\left(\frac{d}{dt}\right)^n M_X(t) = E[X^n e^{tX}]$ for $|t| < \varepsilon$.

$\therefore M_X^{(n)}(0) = E[X^n]$. $\Leftarrow$ moments of X.

Actually, we can do this all at once. Note: $e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n$.

Prop: If $e^{\varepsilon|X|} \in L^1$ then M_X is analytic on $(-\varepsilon, \varepsilon)$ and

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n] \qquad |t| < \varepsilon.$$

Pf. Let $f_N(t,\omega) = \sum_{n=0}^{N} \frac{t^n}{n!} X(\omega)^n$

$$\therefore |f_N(t,\omega)| \le \sum_{n=0}^{N} \frac{|t|^n}{n!} |X(\omega)|^n \le \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} |X(\omega)|^n = e^{\varepsilon|X|}(\omega) \in L^1.$$

$$f_N(t,\cdot) \to e^{tX} \text{ on } \Omega$$

$\therefore$ by DCT $\quad \mathbb{E}[f_N(t,\cdot)] \to \mathbb{E}[e^{tX}] = M_X(t)$

$$\mathbb{E}\left[\sum_{n=0}^{N} \frac{t^n}{n!} X^n\right] = \sum_{n=0}^{N} \frac{t^n}{n!} \mathbb{E}[X^n].$$

$$\therefore \infty > \mathbb{E}[e^{tX}] = \lim_{n\to\infty} \left(\quad " \quad " \quad \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n].$$

Caution:

It's really important that the conditions of the differentiate-under-an-integral proposition hold a.s. independently of t.

Eg. Lebesgue measure $([0,1], \mathcal{B}([0,1]), \lambda)$

$$f(t,\omega) = \mathbb{1}_{\omega \le t} = \mathbb{1}_{[0,t]}(\omega).$$

1. For fixed t, $\mathbb{1}_{[0,t]}$ ✓ measurable.
2. $|f(t,\cdot)| \le 1$ and $\lambda([0,1]) = 1 < \infty$ so $f(t,\cdot) \in L^1$
3. $\frac{\partial f}{\partial t}(\omega) = \frac{\partial}{\partial t}\begin{cases} 1, & t \ge \omega \\ 0, & t < \omega \end{cases} = \begin{cases} 0 & \text{if } t \ne \omega \\ \nexists & \text{if } t = \omega \end{cases} = 0$ a.s.
4. $g \equiv 0 \in L^1$. ✓

$$\frac{d}{dt}\int \mathbb{1}_{[0,t]}\, d\lambda = \frac{d}{dt}\int f(t,\omega)\,\lambda(d\omega) = \int \frac{\partial f}{\partial t}\, d\lambda = \int 0\, d\lambda = 0.$$

$$= \frac{d}{dt}\lambda[0,t] = \frac{d}{dt} t = 1.$$

$\times$