1.2 Algebraic Structures of Subsets (II.4 in Driver)

Start with a sample space Ω (any set).

Definition: A collection $\mathcal{F} \subseteq 2^{\Omega}$ is a <u>field</u> if

1. $\Omega \in \mathcal{F}$
2. If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$. $*$
3. If $E_1, \dots, E_n \in \mathcal{F}$, then $\bigcup_{j=1}^{n} E_j \in \mathcal{F}$ $*$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \underbrace{\bigcup_{i=1}^{n} E_i^c}_{\in \mathcal{F}}$$

$E_i^c \in \mathcal{F}$

If, instead of 3, we have the stronger

3'. If $\{E_j\}_{j=1}^{\infty}$ is a countable set of events in $\mathcal{F}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{F}$

then we call $\mathcal{F}$ a <u>σ-field.</u>

Examples

E.g. $\mathcal{F} = 2^{\Omega}$

E.g. $\mathcal{F} = \{\emptyset, \Omega\}$

E.g. $\Omega = \{1,2,3\}$ $\quad \mathcal{F} = \{\emptyset, \{1\}, \{2,3\}, \Omega\}$

E.g. $\mathcal{F} = \{B \subseteq \Omega : B \text{ is countable or } B^c \text{ is countable}\}$

Lemma: If I is any index set and $\{\mathcal{F}_i : i \in I\}$ are σ-fields over Ω, then $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-field. $\mathcal{F} := \bigcap_i \mathcal{F}_i$

Pf. 1. $\Omega \in \bigcap_i \mathcal{F}_i$ $\quad \Omega \in \mathcal{F}_i \ \forall i \Rightarrow \Omega \in \mathcal{F}$ ✓

2. $E \in \bigcap_i \mathcal{F}_i = \mathcal{F} \Rightarrow E \in \mathcal{F}_i \ \forall i \Rightarrow E^c \in \mathcal{F}_i \ \forall i \Rightarrow E^c \in \bigcap_i \mathcal{F}_i = \mathcal{F}$ ✓

3. $\{E_n\}_{n=1}^{\infty} \in \bigcap_i \mathcal{F}_i \Rightarrow E_n \in \mathcal{F}_i \quad \forall n, i$

$\Rightarrow \bigcup_n E_n \in \mathcal{F}_i \quad \forall i \Rightarrow \bigcup_n E_n \in \mathcal{F}$ ✓

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Prop: Let $\mathcal{E} \subseteq 2^{\Omega}$ be any collection of subsets of Ω.
There is a unique smallest σ-field $\sigma(\mathcal{E})$ that contains $\mathcal{E}$. It is called the σ-field generated by $\mathcal{E}$.

Pf. $\sigma(\mathcal{E}) = \bigcap \{ \mathcal{F} : \mathcal{F}$ is a σ-field over Ω s.t. $\mathcal{E} \subseteq \mathcal{F} \}$.

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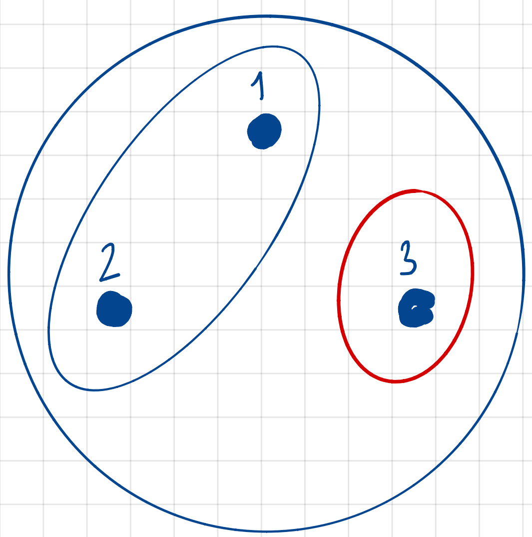
E.g. $\Omega = \{1, 2, 3\}$, $\mathcal{E} = \{\phi, \{1, 2\}, \Omega\}$



$\sigma(\mathcal{E}) = \{\phi, \{1,2\}, \{3\}, \Omega\}$

$\boxed{\text{Exercise}}$ Let $\mathcal{E}_1, \mathcal{E}_2 \subseteq 2^{\Omega}$. Show that
$\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$ iff: $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ & $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$.

The Borel σ-Field

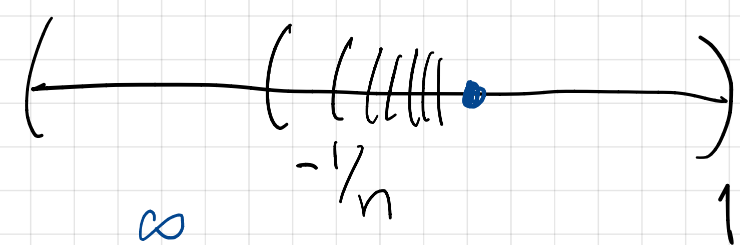
Let X be a topological space (e.g. Euclidean space $\mathbb{R}^d$).
The **Borel σ-field** $\mathcal{B}(X)$ is the σ-field generated by the open subsets in X.

$$\mathcal{B}(X) = \sigma\{\text{open subsets of } X\}$$

Events in $\mathcal{B}(X)$ are called **Borel sets**.

Fun Fact ! : For $X = \mathbb{R}^d$, every open set U is a countable union of open balls

$$U = \bigcup_{i=1}^{\infty} B(x_i, r_i)$$



$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, 1\right) = [0,1)$$

$\therefore \; \mathcal{B}(\mathbb{R}^d) = \sigma\{\text{open balls in } \mathbb{R}^d\}$

$d=1$: $\mathcal{B}(\mathbb{R}) = \sigma\{(a,b) : a<b\} = \sigma\{[a,b) : a<b\}$

$= \sigma\{(a,b] : a<b\}$