1.1 Probability Motivation (I.3 in Driver)

"Experiments"

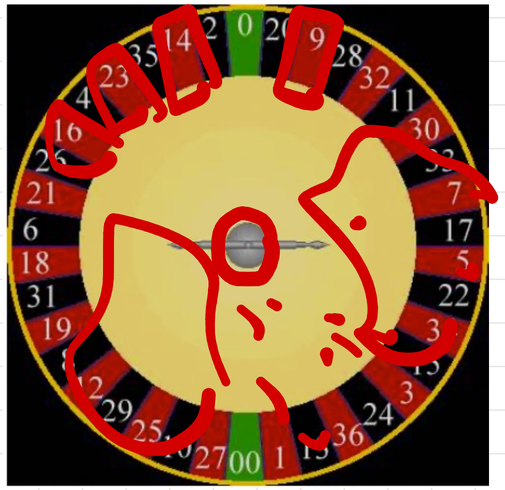
Eg. Toss a fair coin N times.

H T H H

"3 heads, 1 tail"

{HHHT, HHTH, HTHH, THHH}

Eg. Throw a dart at a board of radius R.



- Each experiment has an **outcome**.
- The set of all possible outcomes is the **sample space** Ω.
- "Probability" is a measure of the likelihood of a **set of outcomes** = an **event** $E \subseteq \Omega$.

Finite vs. Countable

Eg. Continue tossing a fair coin until tails comes up. What is the probability that the total number of tosses was odd?

$$\Omega = \{ (\omega_1, \omega_2, \omega_3, \ldots) : \omega_j \in \{H, T\} \}$$
$$= \{ \omega : \mathbb{N} \to \{H, T\} \}$$

$$E = \{ \omega(1) = T \} \sqcup \{ \omega(1) = \omega(2) = H, \ \omega(3) = T \} \sqcup \{ \omega(1) = \cdots = \omega(4) = H, \ \omega(5) = T \} \sqcup \cdots$$

$$= \bigsqcup_{j=0}^{\infty} \{ \omega : \mathbb{N} \to \{H, T\} : \omega(1) \cdots = \omega(2j) = H, \ \omega(2j+1) = T \}$$

$$= \bigsqcup_{j=0}^{\infty} E_j \qquad \mathbb{P}(E_j) = \frac{1}{2^{2j+1}}$$

$$\mathbb{P}(E) = \sum_{j=0}^{\infty} \mathbb{P}(E_j) = \sum_{j=0}^{\infty} \frac{1}{2^{2j+1}} = \frac{2}{3}.$$

Putative Definition

Let Ω be a sample space.

A **probability measure** on Ω is a function

$$\mathbb{P} : \cancel{2^{\Omega}} \to [0,1]$$

s.t. (1) $\mathbb{P}(\Omega) = 1$

(2) If $\{E_j\}_{j=1}^{\infty}$ in 2^{Ω} are disjoint, then

$$\mathbb{P}\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

E.g. $E \subseteq \Omega$, $E^c = \Omega \setminus E$ $\quad \therefore \quad E \cup E^c = \Omega$ $\quad (1)$

$$\therefore \quad \mathbb{P}(E \cup E^c) = \mathbb{P}(\Omega) = 1$$

$$\overset{\shortparallel}{\mathbb{P}(E) + \mathbb{P}(E^c)} \quad \therefore \quad \mathbb{P}(E^c) = 1 - \mathbb{P}(E)$$

In particular, $\mathbb{P}(\emptyset) = 1 - 1 = \underline{0}$