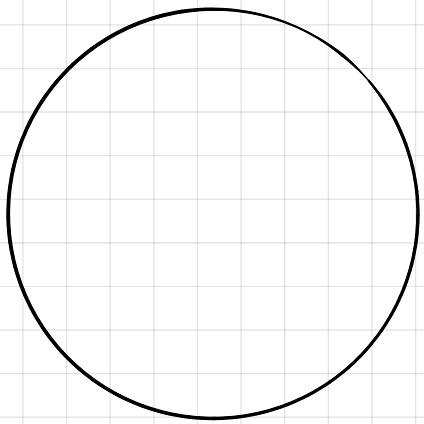
Measurement

To keep things simple, let's talk about measuring angle segments in the unit circle

$$\mathbb{S} = \{ z \in \mathbb{C} : |z| = 1 \}$$



For a given subset $E \subseteq \mathbb{S}$ we'd like to assign a number

$$\mathbb{P}(E) =$$

Properties of Measurements

1. $\mathbb{P}(S) =$, $\mathbb{P} : 2^S \longrightarrow$ ______

2. If $E_1, E_2 \subseteq S$ are ______
   then $\mathbb{P}(E_1 \cup E_2) =$

2'. If $\{E_j\}_{j=1}^{\infty}$ are ______
   then $\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) =$

3. If $E_1, E_2 \subseteq S^1$ are ______
   then $\mathbb{P}(E_1)$ $\mathbb{P}(E_2)$

Theorem:

Proof. If $E \subseteq \mathbb{S}$ and $u \in \mathbb{S}$ then E and

$$uE =$$

are congruent. $\therefore$ by (3), $P(E) = P(uE)$ $\forall u \in \mathbb{S}$.

Now, consider $\mathbb{S} \supset \mathbb{T} := \{ e^{2\pi i t} : t \in \mathbb{Q} \}$ (countable)

$\mathbb{S}/\mathbb{T} = \{$ equivalence classes in $\mathbb{S}$
where $z \sim w \iff z = uw$ for some $u \in \mathbb{T} \}$.

Choose exactly 1 representative element φ from each equivalence class, and let $\Phi = \{\varphi\} \subset \mathbb{S}$ be the collection of all these representatives.

Claim: $\mathbb{S} = \bigsqcup_{u \in \mathbb{T}} u\Phi$.

Contradicting Calculus?

The measurement function $\mathbb{P}$, satisfying (1), (2′), (3) is used daily in Calculus!

$$\mathbb{P}(E) =$$

So how can it fail to exist?

The answer lies in an important subtlety: the definition of the Riemann integral only works over "nice" sets. The set Φ is not nice!

Much of this quarter will be spent extending the Riemann integral. BUT there's only so far it can be extended.

The Moral of the Story

$$\mathbb{P} : 2^{\mathcal{S}} \to [0,1]$$

This might seem like a bad sign...
but it is actually a foundational truth for Kolmogorov's probability theory (that we now embark on developing).

In short: we don't always have complete information about the world, which means there may be some events we simply cannot assign probabilities to.

As to the unmeasurable sets...

Banach-Tarski Paradox (1942)

Given any two subsets $E, F \subseteq \mathbb{R}^3$ (or $\mathbb{R}^d$ for $d \geq 3$) with nonempty interior, there are finite partitions

$$E = E_1 \cup E_2 \cup \cdots \cup E_n$$

$$F = F_1 \cup F_2 \cup \cdots \cup F_n$$

such that E_j is congruent to F_j for $1 \leq j \leq n$.

Robinson's Doubling Theorem (1947)

If E is a solid ball in $\mathbb{R}^3$, and F is two disjoint balls of the same radius, then Banach-Tarski works explicitly with $n = 5$.