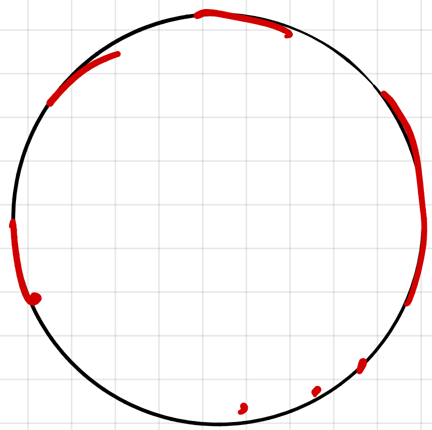
Measurement

To keep things simple, let's talk about measuring angle segments in the unit circle

$$S = \{ z \in \mathbb{C} : |z| = 1 \}$$



For a given subset $E \subseteq S$ we'd like to assign a number

$P(E) =$ proportion of the circle $\in [0, 1]$

Properties of Measurements

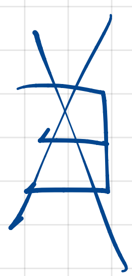
1. $\mathbb{P}(S) = 1$, $\mathbb{P} : 2^S \to [0,1]$

2. If $E_1, E_2 \subseteq S$ are disjoint $E_1 \cap E_2 = \emptyset$
   then $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$

2'. If $\{E_j\}_{j=1}^{\infty}$ are disjoint $E_i \cap E_j = \emptyset \ \forall i \neq j$.
   then $\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$

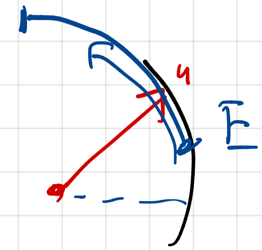
3. If $E_1, E_2 \subseteq S^1$ are congruent
   then $\mathbb{P}(E_1) = \mathbb{P}(E_2)$

Theorem:



Proof. If $E \subseteq \mathbb{S}$ and $u \in \mathbb{S}$ then E and

$u = e^{i\theta}$ $\quad \to uE = \{ uz : z \in E \}$



are congruent. $\therefore$ by (3), $P(E) = P(uE)$ $\forall u \in \mathbb{S}$.

Now, consider $\mathbb{S} \supset \mathbb{T} := \{ e^{2\pi i t} : t \in \mathbb{Q} \}$ (countable)

$\mathbb{S}/\mathbb{T} = \{$ equivalence classes in $\mathbb{S}$ where $z \sim w \iff z = uw$ for some $u \in \mathbb{T} \}$.

Axiom of Choice.

Choose exactly 1 representative element φ from each equivalence class, and let $\Phi = \{\varphi\} \subset \mathbb{S}$ be the collection of all these representatives.

Claim: $\mathbb{S} = \bigsqcup_{u \in \mathbb{T}} u\Phi$.

(1) $\mathbb{S} = \bigcup_{u \in \mathbb{T}} u\Phi$ ✓

(2) if $u_1 \neq u_2 \in \mathbb{T}$, $u_1\Phi \cap u_2\Phi = \emptyset$

$$1 \underset{(1)}{=} P(\mathbb{S}) = P\Big(\bigsqcup_{u \in \mathbb{T}} u\Phi\Big) \underset{(2')}{=} \sum_{u \in \mathbb{T}} P(u\Phi) \underset{(3)}{=} \sum_{u \in \mathbb{T}} P(\Phi) = \begin{cases} 0 \\ \infty \end{cases}$$ ↯

Contradicting Calculus?

The measurement function $\mathbb{P}$, satisfying (1), (2′), (3) is used daily in Calculus!

$$\mathbb{P}(E) = \frac{1}{2\pi}\int_E d\theta$$

$$\int_{(\;)}$$

cannot plug in any old set.

So how can it fail to exist?

The answer lies in an important subtlety: the definition of the Riemann integral only works over "nice" sets. The set Φ is not nice!

Much of this quarter will be spent extending the Riemann integral. BUT there's only so far it can be extended.

The Moral of the Story

$$P : \cancel{2^{S}} \to [0,1]$$

$$\mathcal{F} \subsetneq 2^{S}$$

This might seem like a bad sign...
but it is actually a foundational truth for Kolmogorov's probability theory (that we now embark on developing).

In short: we don't always have complete information about the world, which means there may be some events we simply cannot assign probabilities to.

As to the unmeasurable sets...

Banach-Tarski Paradox (1942)

Given any two subsets $E, F \subseteq \mathbb{R}^3$ (or $\mathbb{R}^d$ for $d \geq 3$) with nonempty interior, there are finite partitions

$$E = E_1 \cup E_2 \cup \cdots \cup E_n$$

$$F = F_1 \cup F_2 \cup \cdots \cup F_n$$

such that E_j is congruent to F_j for $1 \leq j \leq n$.

Robinson's Doubling Theorem (1947)

If E is a solid ball in $\mathbb{R}^3$, and F is two disjoint balls of the same radius, then Banach-Tarski works explicitly with $n = 5$.