MATH 180A - INTRODUCTION TO PROBABILITY
PRACTICE MIDTERM #1

FALL 2019

Name (Last, First): _____

Student ID: _____

> **REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.**

> **THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS ON THE PAPER PROVIDED. DO NOT REMOVE ANY OF THE PAGES.**

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Suppose that $A, B \in \mathcal{F}$ satisfy
$$\mathbb{P}(A) + \mathbb{P}(B) > 1.$$
Making no further assumptions on A and B, prove that $A \cap B \neq \emptyset$.

Sol. 1: Assume that $A \cap B = \emptyset$. Then

$$P(A \cup B) = P(A) + P(B) > 1, \text{ contradiction.}$$

$$A \cap B \neq \emptyset$$

Sol. 2: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$

$$P(A \cap B) \geq P(A) + P(B) - 1 > 0$$

$$\Rightarrow A \cap B \neq \emptyset$$

(b) Prove that A is independent from itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.

A indep. of A if $P(A \cap A) = P(A) \cdot P(A)$

$$\parallel$$

$$P(A)$$

$$\left(P(A)\right)^2 - P(A) = 0 \quad \Rightarrow \quad P(A) \in \{0, 1\}$$

2. Roll two fair dice repeatedly. If the sum is ≥ 10, then you win.

(a) What is the probability that you start by winning 3 times in a row?

$$p = P(Success) = P(\text{roll 2 dice, get} \geq 10)$$
$$= \frac{1}{36} \#\{(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)\}$$
$$= \frac{1}{6}$$

$X \sim B(3, \frac{1}{6})$ gives # of successes after 3 games

$$P(X = 3) = \binom{3}{3} p^3 (1-p)^0 = \frac{1}{6^3}$$

(b) What is the probability that after rolling the pair of dice 5 times you win exactly 3 times?

$Y \sim B(5, \frac{1}{6})$ gives # successes in 5 games

$$P(Y = 3) = \binom{5}{3} p^3 (1-p)^2 = 10 \cdot \frac{1}{6^3} \cdot \left(\frac{5}{6}\right)^2 = \frac{250}{6^5}$$

(c) What is the probability that the first time you win is before the tenth roll (of the pair), but after the fifth?

$Z \sim Geom(\frac{1}{6})$ describes first winning game

$$P(6 \leq Z \leq 9) = P(Z=6) + P(Z=7) + P(Z=8) + P(Z=9)$$
$$= (1-p)^5 p + (1-p)^6 p + (1-p)^7 p + (1-p)^8 p$$
$$= (1-p)^5 p \left(1 + (1-p) + (1-p)^2 + (1-p)^3\right)$$
$$= (1-p)^5 \not{p} \; \frac{1 - (1-p)^4}{1 - (1-p)} = (1-p)^5 - (1-p)^9$$

3. A box contains 3 coins, two of which are fair and the third has probability 3/4 of coming up heads. A coin is chosen randomly from the box and tossed 3 times.

    (a) What is the probability that all 3 tosses are heads?

$A = \{\text{three heads}\}$ $B_1 = \{\text{chosen coin is fair}\}$

$P(A|B_1) = \left(\dfrac{1}{2}\right)^3$ $B_2 = \{\text{chosen cois is biased}\}$

$P(A|B_2) = \left(\dfrac{3}{4}\right)^3$ $P(B_1) = \dfrac{2}{3}$ $P(B_2) = \dfrac{1}{3}$

$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$

$\qquad = \dfrac{1}{8} \cdot \dfrac{2}{3} + \left(\dfrac{3}{4}\right)^3 \cdot \dfrac{1}{3} = \cdots$

$\qquad = \dfrac{43}{192}$

    (b) Given that all three tosses are heads, what is the probability that the biased coin was chosen?

$P(B_2|A) = \dfrac{P(A|B_2)\cdot P(B_2)}{P(A)} = \dfrac{\left(\frac{3}{4}\right)^3 \cdot \frac{1}{3}}{\dfrac{43}{192}} = \dfrac{27}{43}$

4. Let X be a discrete random variable taking the values $\{1, 2, \ldots, n\}$ all with equal probability. Let Y be another discrete random variable taking values in $\{1, 2, \ldots, n\}$. Assume that X and Y are independent. Show that $\mathbb{P}(X = Y) = \frac{1}{n}$. (Hint: you do not need to know the distribution of Y to calculate this.)

$$P(X=Y) = \sum_{k=1}^{n} P(X=Y \mid Y=k) \cdot P(Y=k)$$
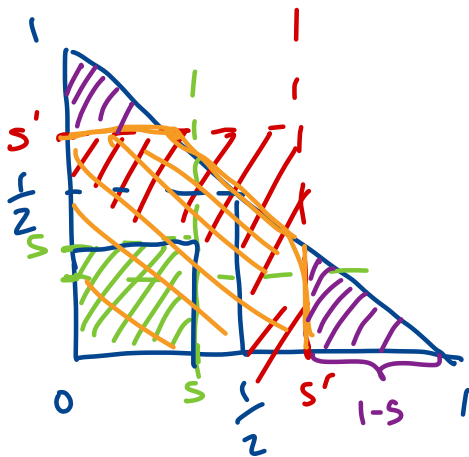
$$= \sum_{k=1}^{n} \frac{P(X=Y, Y=k) \cdot P(Y=k)}{P(Y=k)}$$

can start here $\longrightarrow$

$$= \sum_{k=1}^{n} P(X=k, Y=k) = \sum_{k=1}^{n} \underbrace{P(X=k)}_{=\frac{1}{n}} P(Y=k)$$

$$= \frac{1}{n} \sum_{k=1}^{n} P(Y=k) = \frac{1}{n}$$

5. Consider a point $P = (X, Y)$ chosen uniformly at random inside of the triangle in $\mathbb{R}^2$ that has vertices $(1, 0)$, $(0, 1)$, and $(0, 0)$. Let $Z = \max(X, Y)$ be the random variable defined as the maximum of the two coordinates of the point. For example, if $P = (\frac{1}{2}, \frac{1}{3})$, then $Z = \max(X, Y) = \frac{1}{2}$. Determine the cumulative distribution function of Z. Determine if Z is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

(Hint: Draw a picture.)

$$Z = \max(X, Y)$$

$$\{Z \le s\} = \{\max(X, Y) \le s\}$$
$$= \{X \le s \text{ and } Y \le s\}$$

$$P\{Z \le s\} = \begin{cases} \dfrac{s^2}{\frac{1}{2}}, & 0 \le s \le \frac{1}{2} \\[2mm] \dfrac{\frac{1}{2} - 2 \cdot \frac{(1-s)^2}{2}}{\frac{1}{2}}, & 1\frac{1}{2} \le s \le 1 \end{cases}$$

$$F_X(s) = \begin{cases} 0, & s < 0 \\[1mm] 2s^2, & 0 \le s < \frac{1}{2} \\[1mm] 1 - 2(1-s)^2, & \frac{1}{2} \le s < 1 \\[1mm] 1, & s \ge 1 \end{cases}$$

$$\lim_{s \to \frac{1}{2}^-} F_X(s) = 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

$$F_X\left(\tfrac{1}{2}\right) = 1 - 2\left(1 - \tfrac{1}{2}\right)^2 = \tfrac{1}{2}, \quad X \text{ continuous}$$

$$f_X(s) = \begin{cases} 0, & s \le 0 \\[1mm] 4s, & 0 < s \le \frac{1}{2} \\[1mm] 4 - 4s, & \frac{1}{2} < s \le 1 \\[1mm] 0, & s > 1 \end{cases}$$

MATH 180A - INTRODUCTION TO PROBABILITY
PRACTICE MIDTERM #1 ADDITIONAL PROBLEMS

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1. At a political meeting there are 7 progressives and 7 conservatives. We choose five people uniformly at random to form a committee (president, vice-president and 3 regular members).

    (a) Let A be the event that we end up with more conservatives than progressives. What is the probability of A?

    (b) Let B be the event that Ronald, representing conservatives, becomes the president, and Felix, representing liberals, becomes the vice-president. What is the probability of B?

(a) $A_i = \{i \text{ conservatives in the committee}\}$

$$P(A_i) = \frac{\binom{7}{i}\binom{7}{5-i}}{\binom{14}{5}}$$

$B = \{\text{more conservatives than progressives}\}$

$B = A_3 \cup A_4 \cup A_5$

$$P(B) = \frac{\binom{7}{3}\binom{7}{2} + \binom{7}{4}\binom{7}{1} + \binom{7}{5}\binom{7}{0}}{\binom{14}{5}}$$

(b) $C = \{\text{Ronald president, Felix vice-president}\}$

$$P(C) = \frac{\cancel{\binom{12}{3}}}{\cancel{\binom{14}{5}}} = \cancel{\frac{\frac{12!}{9}}{\frac{14!}{5 \cdot 9!}}} = \cancel{\frac{9 \cdot 5}{13 \cdot 14}} = \cancel{\frac{12}{5 \cdot 2}}$$

Use order: $(\text{pres, vice pres, reg, reg, reg})$

Total # outcomes: $14 \cdot 13 \cdot 12 \cdot 11 \cdot 10$

Favorable outcomes: $1 \cdot 1 \cdot 12 \cdot 11 \cdot 10$

$$P(C) = \frac{12 \cdot 11 \cdot 10}{14 \cdot 13 \cdot \cancel{12 \cdot 11 \cdot 10}} = \frac{1}{14 \cdot 13}$$

2. Let A, B be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Suppose that A, B satisfy
$$\mathbb{P}(A) + \mathbb{P}(B) > 1.$$
Making no further assumptions on A and B, prove that $A \cap B \neq \emptyset$.

(b) Suppose that A, B satisfy $A \cap B = \emptyset$. If A and B are independent, what can you say about $\mathbb{P}(A)$ and $\mathbb{P}(B)$?

(c) Suppose that $\mathbb{P}(A) = 0.5$ and $\mathbb{P}(B) = 0.8$. What possible range of values can $\mathbb{P}(A \cap B)$ have?

(a) Similar as 1(a) in Practice midterm

(b) If A and B independent, then

$$P(A \cap B) = P(A) P(B) = P(\emptyset) = 0$$

$\Rightarrow$ either $P(A) = 0$ or $P(B) = 0$

(c) (i) $0 \leq P(A \cup B) \leq 1$

$0 \leq P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow P(A \cap B) \geq P(A) + P(B) - 1 = 0.3$

(ii) $A \cap B \subset A$, $A \cap B \subset B$

$\Rightarrow P(A \cap B) \leq P(A) = 0.5$, $P(A \cap B) \leq P(B) = 0.8$

(i), (ii) $\Rightarrow$ $0.3 \leq P(A \cap B) \leq 0.5$

3. Suppose that we have two possibly unfair coins: the first coin takes heads with probability $p \in (0,1)$ and tails with probability $1 - p$; the second coin takes heads with probability $q \in (0,1)$ and tails with probability $1 - q$. The coins are independent of each other and consecutive flips are also independent.

Consider the following game. Flip both coins simultaneously. If the coins land on the same side (for example, both land on heads), then you win. Otherwise, then you lose. Let X be the number of times you play the game until you win. For example, X is equal to 1 if you win on your first play of the game. Determine the probability mass function of X.

$$P(\text{Win}) = P\{(H,H)\} + P\{(T,T)\} = pq + (1-p)(1-q)$$

$$= 1 + 2pq - p - q =: s$$

$$X \sim \text{Geom}(s)$$

$$P(X = k) = (1-s)^{k-1} s = (p + q - 2pq)^{k-1} (1 + 2pq - p - q)$$

4. Consider a point $P = (X, Y)$ chosen uniformly at random inside of the unit square $[0,1]^2 = [0,1] \times [0,1] = \{(x,y) : 0 \le x, y \le 1\}$. Let $Z = \min(X, Y)$ be the random variable defined as the minimum of the two coordinates of the point. For example, if $P = (\frac{1}{2}, \frac{1}{3})$, then $Z = \min(\frac{1}{2}, \frac{1}{3}) = \frac{1}{3}$. Determine the cumulative distribution function of Z. Determine if Z is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

(Hint: Draw a picture.)
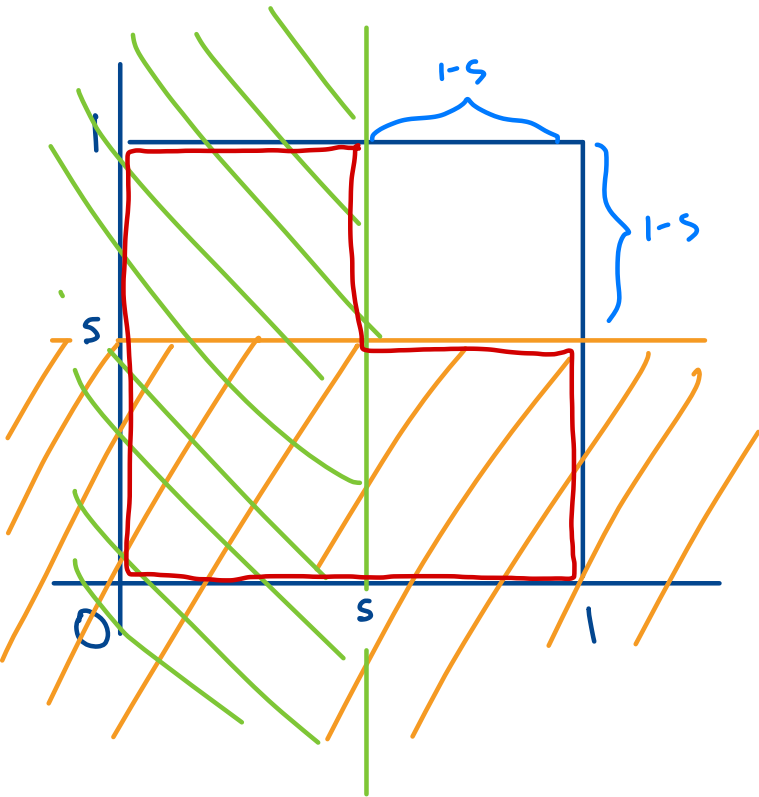
$$\{\min(X,Y) \le s\}$$

$$= \{\text{either } X \le s \text{ or } Y \le s\}$$

$$= \{X \le s\} \cup \{Y \le s\}$$

$$\text{Area } = 1 - (1-s)^2$$

$$= 2s - s^2$$

$$F_X(s) = \begin{cases} 0, & s < 0 \\ 2s - s^2, & 0 \le s < 1 \\ 1, & s \ge 1 \end{cases}$$
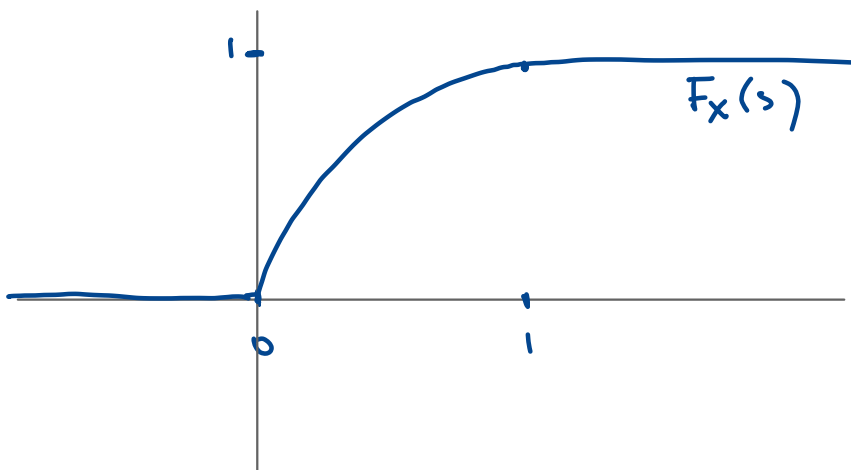
$$f_X(s) = \begin{cases} 0, & s < 0 \\ 2 - 2s, & 0 \le s < 1 \\ 1, & s \ge 1 \end{cases}$$

5. You shoot an arrow (uniformly at random) at a round target of radius 50 cm. If you hit a point at a distance ≤ 10 cm from the center of the target, you are awarded 10 points; if the point you hit is between 10 and 20 cm from the center, you get 5 point; if the point you hit is between 20 and 30 cm from the center, you get 3 points; if the point you hit is between 30 and 40 points from the center you get 1 point; if you hit a point which is 40 cm or more from the center you get 0 points. Let X be a random variable that gives the total number of points you get after ~~two shots~~ one shot.

    (a) Is the random variable X continuous, discrete, neither or both?
    (b) If X is continuous or discrete, compute the p.m.f./p.d.f. of X.
    (c) Compute and plot the c.d.f. of the random variable X.

(a) X is discrete, since it has only finite number of possible outcomes

(b) p.m.f.

k	0	1	3	5	10
$P(X=k)$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{5}{25}$	$\frac{3}{25}$	$\frac{1}{25}$

(c)
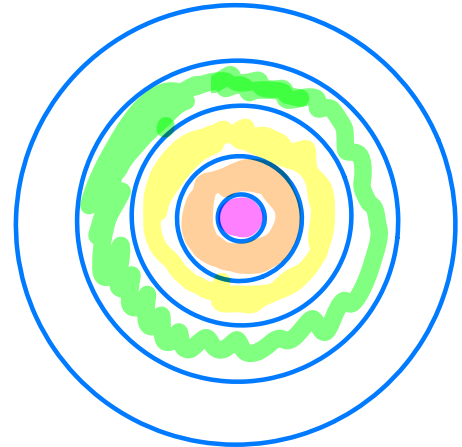
$$F_X(s) = \begin{cases} 0, & s < 0 \\ \frac{9}{25}, & 0 \leq s < 1 \\ \frac{16}{25}, & 1 \leq s < 3 \\ \frac{21}{25}, & 3 \leq s < 5 \\ \frac{24}{25}, & 5 \leq s < 10 \\ 1, & s \geq 10 \end{cases}$$

$F_X(s)$