ID Check? ______

Name: Sol Utions

PID: A$\emptyset$

Seat Number: 1

To affirm your commitment to academic integrity, **write out** the phrase "I excel with integrity"

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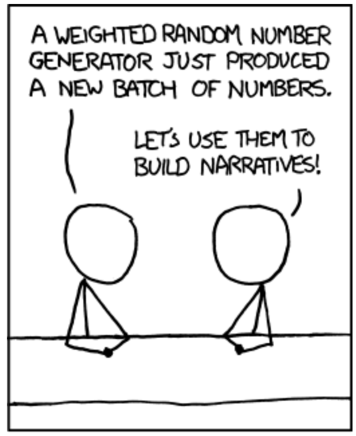

ALL SPORTS COMMENTARY

MATH 180A (B00) EXAM 2

November 20, 2019

INSTRUCTIONS — READ THIS NOW

- Write your name, PID, and current seat number, and write out the academic integrity pledge in the indicated space above. Also, write your name and PID on **every page**. Your score **will not be recorded** if any of this identifying information or the pledge is missing.
- Write within the **boxed areas**, or your work will not be graded. If you need more space, raise your hand. To receive full credit, your answers must be neatly written and logically organized.
- To ask questions during the exam, remain in your seat and raise your hand. Please show your ID to a proctor when you hand in your exam.
- You may not speak to any other student in the exam room while the exam is in progress. You may not share **any information** about the exam with any student who has not yet taken it.
- Turn off and put away all cellphones, calculators, and other electronic devices. You may not access any electronic device during the exam period. You may use your one double-sided page of hand-written notes, but no other books, notes, or aids.
- This is a 50-minute exam in design, but you have up to 2 hours to complete it. There are 5 problems on 5 pages, worth a total of 50 points. Read all the problems first before you start working on any of them, so you can manage your time wisely.
- **Have fun!**

Name: ______________________________ PID: ________________________ **(P. 3)**

1. San Diego gets on average 1 major rain storm per year (defined as a single storm event with more than 5mm of rain at the coast).

(a) (5 points) Using an appropriate Poisson distribution, compute the probability that San Diego will get at least 3 major rain storms this year. (For reference in calculation: $e^{-1} \approx 0.368$.)

$X = \#\text{storms}/\text{year} \sim \text{Poisson}(1)$

$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X < 3) = 1 - \sum_{k=0}^{2} e^{-1} \frac{1^k}{k!}$

$= 1 - e^{-1}(1 + 1 + \frac{1}{2})$

$= 1 - 2.5e^{-1}$

$\doteq 8\%$

(b) (5 points) Let X be the number of major rain storms San Diego gets in a year. Show that the probability calculated in part **(a)** is also equal to $\mathbb{P}(|X - 1| \geq 2)$. What does Chebyshev's inequality say about this probability for a general random variable X? How does it compare to the exact answer you computed above?

$\mathbb{P}(|X-1| \geq 2) = \mathbb{P}(X-1 \geq 2 \text{ or } X-1 \leq -2)$

$= \mathbb{P}(X-1 \geq 2) + \mathbb{P}(X-1 \leq -2)$

$= \mathbb{P}(X \geq 3) + \cancel{\mathbb{P}(X \leq -1)}$

0 b/c $X \geq 0$

$\mathbb{E}(X) = 1$, $\text{Var}(X) = 1$ (Poisson(1))

$\therefore$ Chebyshev: $\mathbb{P}(|X-1| \leq 2) \leq \frac{1}{2^2} = \frac{1}{4}$

Actual value $\doteq 8\% < 25\%$

Name: ______________________ PID: __________________ **(P. 5)**

2. A poll is conducted to determine if UCSD students are in favor of the university administration's decision to spend billions of dollars on fancy new buildings instead of spending money hiring more faculty to enable smaller class sizes. 400 students are polled, and the proportion who support the administration's plans is $\hat{p} = 0.21$.

(a) (5 points) Determine the 95% confidence interval for this poll. (The normal-table-lookup you need is $\Phi(1.96) = 0.975$.)

$$\mathbb{P}(|\hat{p} - p| \le \varepsilon) \ge 2\Phi(2\varepsilon\sqrt{n}) - 1 \underset{\text{want}}{\ge} 0.95$$

$\uparrow$ $n = 400$

$\therefore$ want $\Phi(2\varepsilon\sqrt{400}) \ge 0.975$

want $2\varepsilon \cdot 20 \ge \Phi^{-1}(0.975) = 1.96$

want $\varepsilon \ge \frac{1.96}{40} = 0.049$

$\therefore$ 95% conf. int. $= [0.21 - 0.049, 0.21 + 0.049]$

$= [0.161, 0.259]$

(a) (5 points) The poll organizers want to get a margin of error no more than $\epsilon = 0.02$, with probability 95%, in their next poll. How many students do they need to poll?

Now n unknown, $\varepsilon = 0.02$.

want $\Phi(2\varepsilon\sqrt{n}) \ge 0.975$

i.e. $2(0.02)\sqrt{n} \ge \Phi^{-1}(0.975) = 1.96$

$\sqrt{n} \ge \frac{1.96}{0.04} = 49$

So want $n \ge 49^2 = 2401$

3. Based on past experience, we expect that the number of students who turn in tonight's exam between 9pm and 9:10pm will be about 20.

(a) (5 points) What is the probability that exactly 15 students will turn in the exam between 9pm and 9:10pm?

X = # students $\sim$ Poisson(λ). Find λ.

$\lambda = \mathbb{E}(X) = 20$.

So $\mathbb{P}(X=15) = e^{-20}\dfrac{20^{15}}{15!} \doteq 5.16\%$

(b) (5 points) What is the probability that at least one student will turn in their exam between 9pm and 9:04pm?

T = time until first handed in $\sim$ Exp(λ')

Find λ'.

$\mathbb{E}(T) = 1/\lambda'$ ⟶ $\therefore \frac{1}{\lambda'} = \frac{1}{2}$, so $\lambda' = 2$

20 in 10 minutes $\Rightarrow$ average wait time: $\frac{10}{20}$ min $= \frac{1}{2}$

So

$\mathbb{P}$(at least one by 9:04) $= \mathbb{P}(T \leq 4)$

$= \int_0^4 2e^{-2t}\,dt$

$= -e^{-2t}\Big|_0^4 = 1 - e^{-8} = 99.97\%$

Name: ______________________ PID: ____________________

4. Let $U \sim \text{Unif}[0,1]$ be a uniform random variable, and let $X = 2\ln(1/U)$.

(a) (5 points) Compute the probability density function $f_X(x)$ for all $x \in \mathbb{R}$.

Start w $F_X(t) = \mathbb{P}(X \le t) = \mathbb{P}(2\ln(1/u) \le t)$

$= \mathbb{P}(\frac{1}{u} \le e^{t/2})$

$= \mathbb{P}(u \ge e^{-t/2})$

$= 1 - \mathbb{P}(u < e^{-t/2})$

$= 1 - \begin{cases} 0 & \text{if } e^{-t/2} < 0 \\ e^{-t/2} & \text{if } e^{-t/2} \in [0,1] \\ 1 & \text{if } e^{-t/2} > 1 \end{cases}$ can't happen $(t \ge 0)$ $(t < 0)$

$= \begin{cases} 0 & t < 0 \\ 1 - e^{-t/2} & t \ge 0 \end{cases}.$

$\therefore f_X(t) = \frac{d}{dt}(\text{" "}) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}e^{-t/2} & t \ge 0 \end{cases}.$

I.e. $X \sim \text{Exp}(\frac{1}{2})$

(b) (5 points) Given that $X > 15$, what is the probability that $X > 17$?

Memoryless property of $\text{Exp}(\frac{1}{2})$:

$\mathbb{P}(X > 17 \mid X > 15) = \mathbb{P}(X > 15 + 2 \mid X > 15)$

$= \mathbb{P}(X > 2) = 1 - \mathbb{P}(X \le 2)$

$= 1 - (1 - e^{-2/2})$

$= e^{-1} \doteq 0.368$

5. Let X be a random variable with pdf $f_X(x) = xe^{-x}\mathbb{1}\{x \geq 0\}$.

(a) (7 points) Compute the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$ of X. Make sure to specify the domain on which it is finite. Also: explain how you can tell from the formula you derive for $M_X(t)$ that f_X really is a probability density function.

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tx} \cdot x e^{-x}\,dx = \int_0^\infty x\, e^{(t-1)x}\,dx$$

$\uparrow$ exponential blow up iff $t-1 \geq 0$.

$\therefore M_X(t) = \infty$ if $t \geq 1$.

If $t<1$, $$\int_0^\infty \underbrace{x}_{u}\underbrace{e^{(t-1)x}\,dx}_{dv} = x \cdot \frac{1}{t-1} e^{(t-1)x}\Big|_0^\infty - \int_0^\infty \frac{e^{(t-1)x}}{(t-1)}\,dx$$

$$= 0 - 0 - \left(\frac{1}{(t-1)^2} e^{(t-1)}\Big|_0^\infty\right)$$

$$= \frac{1}{(t-1)^2}.$$

From here, we see $\int_{-\infty}^{\infty} f_X(x)\,dx = M_X(0) = \frac{1}{(0-1)^2} = 1$ ✓

(b) (3 points) Is the distribution of X the *only* probability distribution that has the moment generating function you computed in part **(a)**? Explain your answer.

Yes, M_X uniquely determines the dist. of X, b/c $M_X(t) < \infty$ for t in an interval containing 0 (namely $(-\infty, 1)$).