ID Check? ______

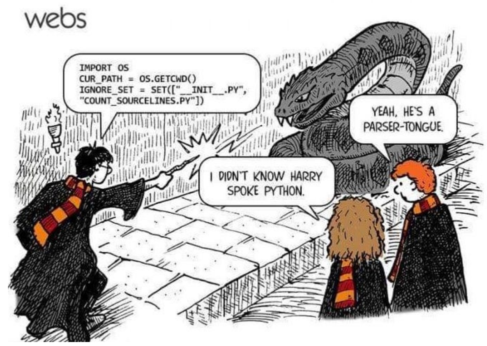
Name: Sol Utions

PID: 42

Seat Number: A∅

To affirm your commitment to academic integrity, **write out** the phrase "I excel with integrity"

I excel with integrity



MATH 180A (B00) EXAM 1

October 23, 2018

INSTRUCTIONS — READ THIS NOW

- Write your name, PID, and current seat number, and write out the academic integrity pledge in the indicated space above. Also, write your name and PID on **every page**. Your score **will not be recorded** if any of this identifying information is missing, or if the academic integrity pledge is illegible.
- Write within the **boxed areas**, or your work will not be graded. If you need more space, raise your hand. To receive full credit, your answers must be neatly written and logically organized.
- To ask questions during the exam, remain in your seat and raise your hand. Please show your ID to a proctor when you hand in your exam.
- You may not speak to any other student in the exam room while the exam is in progress (including after you hand in your own exam). You may not share **any information** about the exam with any student who has not yet taken it.
- Turn off and put away all cellphones, calculators, and other electronic devices. You may not access any electronic device during the exam period. You may use your one double-sided page of hand-written notes, but no other books, notes, or aids.
- This is a 50-minute exam in design, but you have up to 2 hours to complete it. There are 5 problems on 5 pages, worth a total of 50 points. Read all the problems first before you start working on any of them, so you can manage your time wisely.
- **Have fun!**

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Name: Sol Utilons PID: 42

1. Two fair 6-sided dice are rolled.

(a) (3 points) Let $A = \{\text{the first roll is an even number}\}$. What is $\mathbb{P}(A)$?

$\Omega = \{(a,b) : 1 \leq a,b \leq 6\}$ $\#A = 36$

$A = \{(2,b),(4,b),(6,b) : 1 \leq b \leq 6\}$ $\#A = 6+6+6 = 18$

$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{18}{36} = \boxed{\frac{1}{2}}$

(b) (3 points) Let $B = \{\text{the sum of the two dice is 4 or 9}\}$. what is $\mathbb{P}(B)$?

Same Ω as above.

$B = \{(a,b) : a+b = 4 \text{ or } a+b = 9\}$

$= \{(1,3),(2,2),(3,1),(3,6),(4,5),(5,4),(6,3)\}$

$\#B = 7$.

$\therefore \mathbb{P}(B) = \frac{\#B}{\#\Omega} = \boxed{\frac{7}{36}}$

(c) (4 points) Are A and B independent events? Explain why or why not.

$A \cap B = \{(2,2),(4,5),(6,3)\}$.

$\therefore \mathbb{P}(A \cap B) = \frac{3}{36} = \frac{1}{12}$

$\mathbb{P}(A)\mathbb{P}(B) = \frac{1}{2} \cdot \frac{7}{36} = \frac{7}{72} \neq \frac{6}{72} = \frac{1}{12}$

$\therefore$ by def. of independence, A, B are <u>not</u>.

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Name: Sol Utilons PID: 42 **(P. 5)**

2. (10 points) At a Halloween party, a bowl contains three kinds of "fun-sized" candy bars. There are 15 Almond Joy, 20 Butterfinger, and 25 Crunchy bars. You take 10 candy bars out of the jar at random (since you're going to eat them, you do not put them back in). What is the probability that there is at least one kind of candy bar of which you get exactly 5?

Sampling w/out replacement, order doesn't matter.

15+20+25 = 60 candy bars in total, sample 10,

So sample space has $\#\Omega = \binom{60}{10}$.

$\underline{E} = \{\text{at least 1 kind has exactly 5}\}$

$= A \cup B \cup C$ ← exactly 2 Crunchy

A: exactly 2 Almond, B: exactly 2 Butter

$P(E) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + \cancel{P(ABC)}$

$\cancel{P(ABC)}$: $\emptyset$ b/c would need 15 bars out of 10.

$P(A) = \binom{15}{5}\binom{20+25}{5} / \binom{60}{10}$

$P(B) = \binom{20}{5}\binom{15+25}{5} / \binom{60}{10}$

$P(C) = \binom{25}{5}\binom{15+20}{5} / \binom{60}{10}$

$P(AB) = \binom{15}{5}\binom{20}{5} / \binom{60}{10}$

$P(AC) = \binom{15}{5}\binom{25}{5} / \binom{60}{10}$

$P(BC) = \binom{20}{5}\binom{25}{5} / \binom{60}{10}$

$$\boxed{\therefore P(\underline{E}) = \frac{\left\{\binom{15}{5}\binom{45}{5} + \binom{20}{5}\binom{40}{5} + \binom{25}{5}\binom{35}{5} - \binom{15}{5}\binom{20}{5} - \binom{15}{5}\binom{25}{5} - \binom{20}{5}\binom{25}{5}\right\}}{\binom{60}{5}}}$$

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Name: Sol Utions PID: 42 **(P. 7)**

3. Five bags sit on the table in front of you. Each contains 4 coins. One bag has all silver coins; one bag has all gold coins; one has 1 gold and 3 silver; one has 3 gold and 1 silver; and one has 2 of each color. You choose a bag at random and pull a coin out (without looking inside).

(a) (5 points) What is the probability that the coin you chose is gold?

bags B_0, B_1, B_2, B_3, B_4 (B_j has j gold coins)

G = gold coin after selecting bag @ random.

$$\mathbb{P}(G) = \sum_{j=0}^{4} \mathbb{P}(G|B_j)\underbrace{\mathbb{P}(B_j)}_{=1/5} \qquad \mathbb{P}(G|B_j) = \frac{\#\text{gold}}{4} = \frac{j}{4}$$

$$= \frac{1}{5}\left(\frac{0}{4}+\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}\right) = \frac{10}{20} = \boxed{\frac{1}{2}}.$$

(b) (5 points) Given that the coin you chose is gold, what is the probability that the other three coins in the bag you chose are all silver?

Bayes' Rule:

$$\mathbb{P}(\{1 \text{ gold}, 3 \text{ silver}\} | G)$$

$$= \mathbb{P}(B_1|G) = \frac{\mathbb{P}(B_1 G)}{\mathbb{P}(G)} = \frac{\mathbb{P}(G|B_1)\mathbb{P}(B_1)}{\mathbb{P}(G)}$$

$$= \frac{1/4 \cdot 1/5}{1/2} = \boxed{\frac{1}{10}}$$

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Name: Sol Utions PID: 42

4. In an archery class, there are 3 circular targets, with radii 1, 2, and 3 feet. You toss a fair coin twice, and then shoot an arrow at the target of radius 1 + the number of tails that came up; it hits a uniformly random point on the target, with distance Y from the center.

(a) (5 points) What is the cumulative distribution function $F_Y(r)$ for $0 \le r \le 1$?

D_j = shoot @ target of radius j.

$\mathbb{P}(D_j) = \mathbb{P}(\#\text{heads in 2 tosses} = j+1)$

$\mathbb{P}(D_1) = \mathbb{P}(\text{no heads}) = 1/4$

$\mathbb{P}(D_2) = \mathbb{P}(1\text{ head}) = 1/2$

$\mathbb{P}(D_3) = \mathbb{P}(2\text{ heads}) = 1/4$.

$$F_Y(r) = \mathbb{P}(Y \le r)$$

$$= \sum_{j=1}^{3} \mathbb{P}(Y \le r \mid D_j)\,\mathbb{P}(D_j)$$

$$= \frac{1}{4}\mathbb{P}(Y \le r \mid D_1) + \frac{1}{2}\mathbb{P}(Y \le r \mid D_2) + \frac{1}{4}\mathbb{P}(Y \le r \mid D_3)$$

$$\left(\mathbb{P}(Y \le r \mid D_j) = \frac{\text{Area}(\text{Disk}_r)}{\text{Area}(\text{Disk}_j)} = \frac{\pi r^2}{\pi j^2} = r^2/j^2.\right)$$

$$F_Y(r) = \frac{1}{4}r^2 + \frac{1}{2}\left(\frac{r^2}{2^2}\right) + \frac{1}{4}\left(\frac{r^2}{3^2}\right) = \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{36}\right)r^2 = \frac{29}{72}r^2.$$

(b) (5 points) Does Y have a probability density function? If so, compute it, $f_Y(r)$, for $0 \le r \le 1$.

In $0 \le r \le 1$, F_Y is differentiable, so

$$f_Y(r) = F_Y'(r) = \frac{29}{36}r \qquad 0 \le r \le 1.$$

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Name: Sol Uti'ons PID: 42

5. (10 points) Two fair 6-sided dice are rolled, repeatedly. The first die is rolled until the first 3 comes up. The second die is rolled until the first 6 comes up. What is the probability that the two dice are rolled the same number of times?

N_1 = #rolls until 1st die comes up 3.

N_2 = #rolls until 2nd die comes up 6.

N_1, N_2 both have Geom$(\frac{1}{6})$ distribution.

They are independent.

$$\mathbb{P}(N_1 = N_2) = \sum_{k=1}^{\infty} \mathbb{P}(N_1 = N_2 \mid N_1 = k)\,\mathbb{P}(N_1 = k)$$

independence

$$= \sum_{k=1}^{\infty} \mathbb{P}(N_2 = k \mid N_1 = k)\,\mathbb{P}(N_1 = k)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(N_2 = k)\,\mathbb{P}(N_1 = k)$$

$$= \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$$

$$= \frac{1}{36} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k-2} = \frac{1}{36} \sum_{k=1}^{\infty} \left(\left(\frac{5}{6}\right)^2\right)^{k-1}$$

$$= \frac{1}{36} \frac{1}{1 - \left(\frac{5}{6}\right)^2}$$

$$= \frac{1}{36} \cdot \frac{6^2}{6^2 - 5^2} = \frac{1}{36 - 25} = \boxed{\frac{1}{11}}$$

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