MATH180A: Introduction to Probability

(for data science)

www.math.ucsd.edu/~tkemp/180A

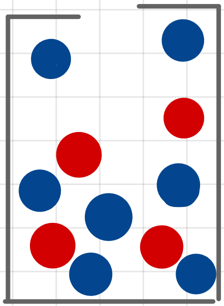
Today: ASV 1.5 and 3.1

Next: ASV 3.2

This week:

homework #2 $\boxed{\text{(due Friday Oct, 11 11:59pm)}}$

E.g. (from last lecture)



An urn has 4 red and 7 blue balls. Choose two balls.

$A = \{1^{st} \text{ ball is red}\}$

$B = \{2^{nd} \text{ ball is blue}\}$

1) choose balls with replacement

$P(A) = \frac{4 \cdot 11}{11 \cdot 11} = \frac{4}{11}$

$P(B) = \frac{11 \cdot 7}{11 \cdot 11} = \frac{7}{11}$

$P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 11} = P(A)\,P(B)$

$\boxed{A \text{ and } B \text{ independent}}$

2) choose balls without replacement

$P(A) = \frac{4 \cdot 10}{11 \cdot 10} = \frac{4}{11}$

$P(B) = \frac{10 \cdot 7}{11 \cdot 10} = \frac{7}{11}$

$P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 10} \neq \frac{4 \cdot 7}{11 \cdot 11}$

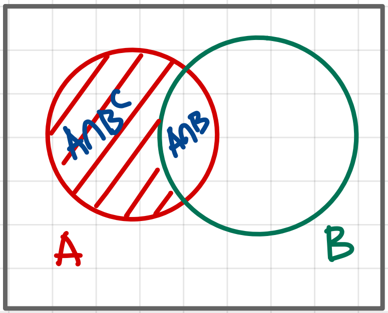
A and B are <u>not</u> independent

A and B independent $\Longleftrightarrow$ A and B^c independent

<u>Proof</u>. ($\Rightarrow$) Suppose that A and B are indep.

$$P(A\cap B^c) = P(A) - P(A\cap B) \overset{\text{indep of } A \,\&\, B}{=} P(A) - P(A)P(B) = P(A)(1-P(B))$$

$$= P(A)P(B^c)$$



$A = (A\cap B) \cup (A\cap B^c)$

disjoint

$P(A) = P(A\cap B) + P(A\cap B^c)$

($\Leftarrow$)

More than two events?

<u>Def.</u> A collection $A_1, \dots, A_n$ of events is **mutually independent** if

<u>for any subcollection</u> $A_{i_1}, A_{i_2}, \dots, A_{i_k}$

$(1 \le i_1 < i_2 < \cdots < i_k \le n)$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

<u>E.g.</u> When $n=3$, this means that we must have

$$P(A_1 \cap A_2) = P(A_1) P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1) P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2) P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$$

Important example

Toss a coin three times

$A = \{$there is exactly 1 Tails in the first two$\}$

$B = \{$there is exactly 1 Tails in the last two$\}$

$C = \{$there is exactly 1 Tails in first and last toss$\}$

$A = \{(H,T,*), (T,H,*)\}$ $B = \{(*,H,T), (*,T,H)\}$

$C = \{(H,*,T), (T,*,H)\}$

$P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C)$

$P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(A \cap C) = P(B \cap C)$

$\hookrightarrow$ A, B, C are pairwise indep.

$A \cap B \cap C = \emptyset$

$P(A \cap B \cap C) = 0$

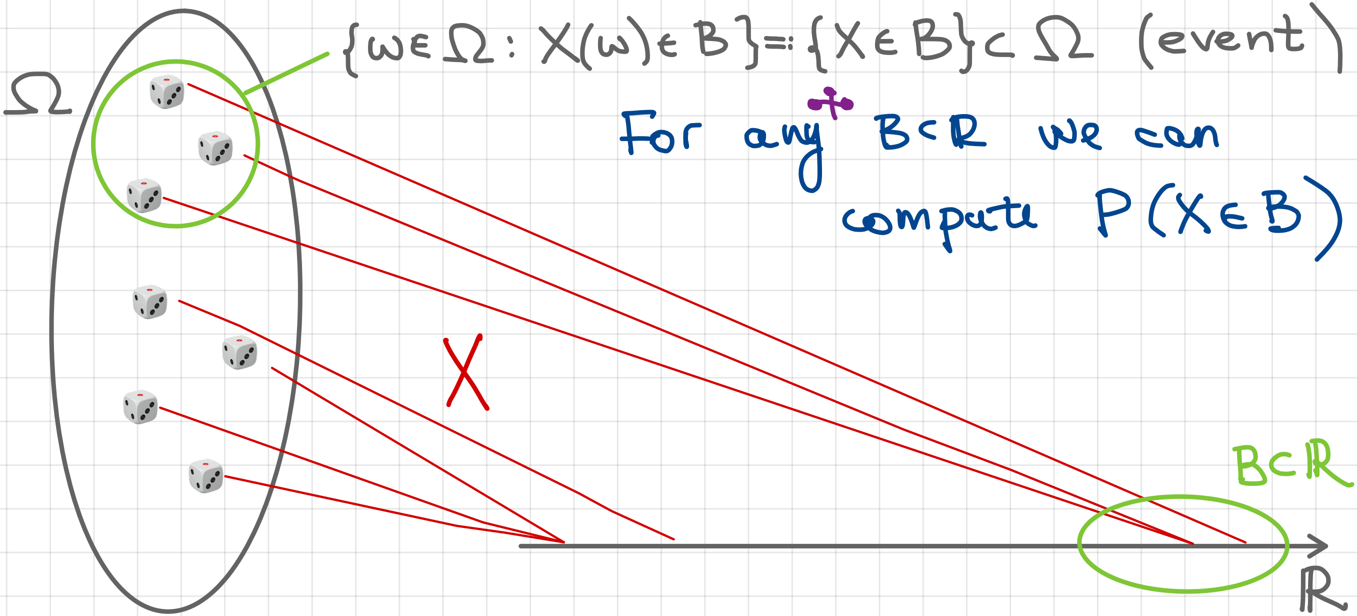
Random variables

$(\Omega, \mathcal{F}, P)$ - probability space

<u>Definition</u>. A (measurable†) function $X : \Omega \to \mathbb{R}$ is called a random variable.

$\{\omega \in \Omega : X(\omega) \in B\} =: \{X \in B\} \subset \Omega$ (event)

For any† $B \subset \mathbb{R}$ we can compute $P(X \in B)$



<u>Def</u>. Let X be a random variable (rv).
The probability distribution of X is the collection of probabilities $P(X \in B)$ for all $B \subset \mathbb{R}$.

<u>Remark</u>. Strictly speaking, $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$
Borel sets

<u>Examples</u> 1) Coin toss : $\Omega = \{H, T\}$, $X(H) = 0$, $X(T) = 1$
$P(X=0) = P(\{H\}) = \frac{1}{2} = P(X=1)$ (fair coin)

2) Roll a die : $\Omega = \{1, 2, \dots, 6\}$, $X(\omega) = \omega$
For any $1 \leq i \leq 6$ $P(X = i) = \frac{1}{6}$

3) Roll a die twice : $\Omega = \{ (i,j) : i,j \in \{1, \dots, 6\} \}$

$X_1((i,j)) = i$ (first number) $X_2((i,j)) = j$ (second number)

for $1 \le i \le 6$ $P(X_1 = i) = \frac{1}{6}$ $P(X_2 = i) = \frac{1}{6}$

$S = X_1 + X_2$

$P(S=2) = \frac{1}{36}$

$P(S=3) = \frac{2}{36}$

$P(S=4) = \frac{3}{36}$

$P(S=5) = \frac{4}{36}$

$P(S=6) = \frac{5}{36}$

$P(S=7) = \frac{6}{36}$

$P(S=8) = \frac{5}{36}$

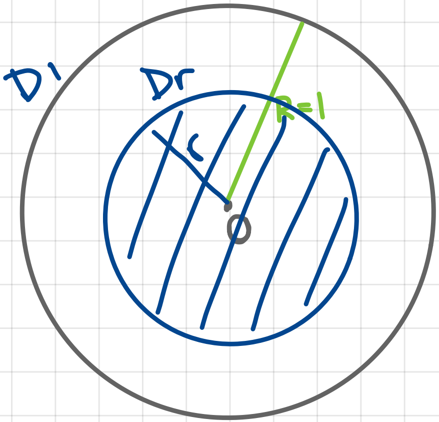
$P(S=9) = \frac{4}{36}$

$P(S=10) = \frac{3}{36}$

$P(S=11) = \frac{2}{36}$

$P(S=12) = \frac{1}{36}$

4) Choosing a point from unit disk unif. at random



$\Omega = \{ w \in \mathbb{R}^2 : dist(w,0) \leq 1 \}$

$X(w) = dist(w,0)$

For any $r<0$, $P(X \leq r) = 0$

For any $r>1$, $P(X \leq r) = 1$

For any $r \in [0,1]$, $P(X \leq r) = \frac{\text{size } D_r}{\text{size } D_1} = \frac{\pi r^2}{\pi} = r^2$

$\{X \leq r\} = \{ X \in (-\infty, r] \}$

missing in class

Def. Random variable X is a discrete rv is there exists a finite or infinite countable collection of points $\{a_1, \dots, \} \subset \mathbb{R}$ such that $\sum_i P(X = a_i) = 1$

Example (lecture 3) Toss a coin until first T.

X = total number of tosses.

(Already computed before) for any $i = 1, 2, \dots$

$$P(X = i) = \frac{1}{2^i}$$

$$\sum_{i=1}^{\infty} P(X = i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \quad \text{(geometric series)}$$

Discrete rv X is completely described by its probability mass function (pmf) p_X

given by

$$p_X(k) = P(X=k)$$

for all possible values of X.

Ex. S = sum of two dice

k	2	3	4	5	6	7	8	9	10	11	12
$p_S(k)$	$\frac{1}{36}$	$\frac{2}{36}$	..	..							

What if for every $x \in \mathbb{R}$ $P(X=x)=0$?

Probability density function

Def. Let X be a rv. If function $f: \mathbb{R} \to \mathbb{R}$ satisfies

$$P(X \le b) = \int_{-\infty}^{b} f(x)\,dx$$

then f is a probability density function of X

Remark. Definition implies that for $B \subset \mathbb{R}$

$$P(X \in B) = \int_{B} f_X(x)\,dx$$

E.g. Distance to 0 from a random point in a disk

$$\int_{-\infty}^{r} f_X(x)\,dx = P(X \le r) = \begin{cases} 0, & r < 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r > 1 \end{cases},$$

$$f_X(x) =$$

Important example : uniform distribution on $[a,b] \subset \mathbb{R}$

Definition Let $a,b \in \mathbb{R}$, $a<b$.

X has uniform distribution on $[a,b]$

(denoted $X \sim \mathcal{U}nif[a,b]$)

if its pdf is given by $f_X(x) = \begin{cases} 0, & x<a \\ \frac{1}{b-a}, & a \le x \le b \\ 0, & x>b \end{cases}$