MATH 180A: INTRO TO PROBABILITY
(FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

Homework 8 Due TODAY

Final Exam: Monday, Dec 9, 11:30a–2:30p
in REC GYM

* Bring Student ID
* Seat Assignment on TritonEd (this weekend)
* 2 double-sided sheets of hand-written notes
* If possible, eat an early lunch.

Regrade requests (for final exam & HW8) NEXT QUARTER.

It is your responsibility to make sure all your grades are in order BEFORE NEXT FRIDAY.

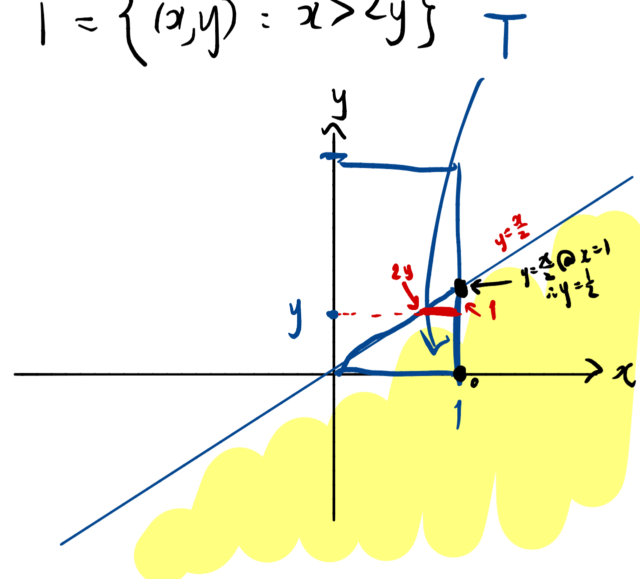
Please fill out your CAPEs!

1. Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{2} + \frac{y}{4}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) (5 points) Compute $\mathbb{P}(X > 2Y)$.

$\{X>2Y\} = \{(X,Y)\in T\} \leftarrow \quad T = \{(x,y) : x > 2y\}$



$$\therefore \mathbb{P}(X>2Y) = \iint_T f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^{1/2} dy \int_{2y}^{1} dx \left(\frac{x}{2}+\frac{y}{4}\right)$$

$$= \int_0^{1/2} \left.\left(\frac{x^2}{4} + \frac{xy}{4}\right)\right|_{2y}^{1} dy$$

$$= \int_0^{1/2} \left(\frac{1-(2y)^2}{4} + \frac{(1-2y)y}{4}\right) dy$$

$$= \frac{1}{4}\int_0^{1/2} (1 - \underbrace{4y^2 + y - 2y^2}_{-6y^2})\,dy = \frac{1}{4}\left(y - 2y^3 + \frac{1}{2}y^2\right)\Big|_0^{1/2}$$

$$= \frac{1}{4}\left(\frac{1}{2} - 2\cdot\left(\frac{1}{2}\right)^3 + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2\right)$$

$$= \frac{1}{4}\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{32}.$$

Calculus

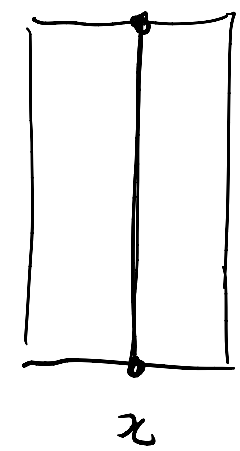
1. Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{2} + \frac{y}{4}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) (5 points) What is the marginal density f_X of X?

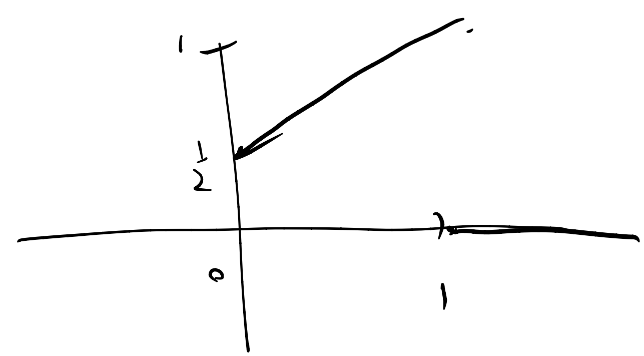
$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy \quad \text{if } 0 \le x \le 1;$$

0 otherwise.



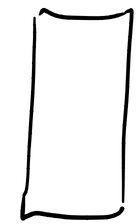
$$= \int_0^2 \left(\frac{x}{2} + \frac{y}{4}\right) dy$$

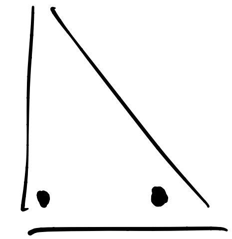
$$= \left(\frac{xy}{2} + \frac{y^2}{8}\right)\Bigg|_{0=y}^{2=y} = \frac{x \cdot 2}{2} + \frac{2^2}{8} = x + \frac{1}{2}.$$



1. Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{2} + \frac{y}{4}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$





(c) (5 points) Are X and Y independent? Explain.

$$f_{(X,Y)}(x,y) = f_X(x)\, f_Y(y) \rightarrow f_Y(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)} = \frac{\frac{x}{2}+\frac{y}{4}}{x+\frac{1}{2}}$$

not a fn of y alone.

No. $\frac{x}{2}+\frac{y}{4}$ is not a product $u(x)v(y)$

S'pose it is. $\dfrac{\frac{x}{2}+\frac{0}{4}}{\frac{x}{2}+\frac{2}{4}} = \dfrac{\cancel{u(x)}\,v(0)}{\cancel{u(x)}\,v(2)}$ $\therefore$ $\dfrac{v(0)}{v(2)} = \dfrac{\frac{x}{2}}{\frac{x}{2}+\frac{1}{2}} = \dfrac{x}{x+1}$

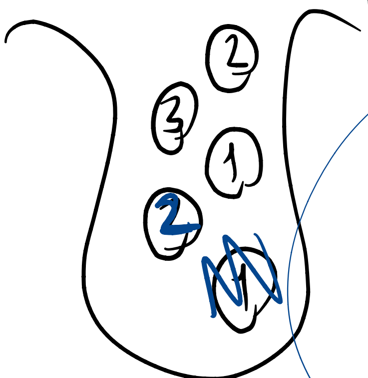
const.

not const.

$\lightning$

2. An urn contains 5 balls: two balls labeled 1, two balls labeled 2, and one ball labeled 3. You sample 2 balls from the bin, without replacement; the number on the first one is X, and the number on the second one is Y.

(a) (5 points) Compute the joint probability mass function of X and Y. (You may find it convenient to express it in the form of a chart.)



1st $\to$ X

2nd $\to$ Y

$P(X=1) = 2/5$

$P(X=2) = 2/5$

$P(X=3) = 1/5$

$$P(X=j, Y=k)$$
$$= P(Y=k \mid X=j)\,P(X=j)$$

$\{X=1\}$ $P(Y=1 \mid X=1) = 1/4$

$P(Y=2 \mid X=1) = 2/4 = 1/2$

$P(Y=3 \mid X=1) = 1/4$

$\{X=3\}$ $P(Y=3 \mid X=3) = 0$

$P((X,Y)=(j,k))$ $1 \le j,k \le 3$

	$X=1$	$X=2$	$X=3$
$Y=1$	$\frac{1}{4}\cdot\frac{2}{5} = 1/10$	$1/5$	$1/10$
$Y=2$	$\frac{1}{2}\cdot\frac{2}{5} = 1/5$	$1/10$	$1/10$
$Y=3$	$\frac{1}{4}\cdot\frac{2}{5} = 1/10$	$1/10$	0

2. An urn contains 5 balls: two balls labeled 1, two balls labeled 2, and one ball labeled 3. You sample 2 balls from the bin, without replacement; the number on the first one is X, and the number on the second one is Y.

(b) (5 points) Compute the probability mass function of X and the probability mass function of Y.

$p_X = p_Y$

	$X=1$	$X=2$	$X=3$	
$Y=1$	$1/4 \cdot 2/5 = 1/10$	$1/5$	$1/10$	$2/5$
$Y=2$	$\frac{1}{2} \cdot 2/5 = 1/5$	$1/10$	$1/10$	$2/5$
$Y=3$	$\frac{1}{4} \cdot 2/5 = 1/10$	$1/10$	0	$1/5$
$p_X \rightarrow$	$2/5$	$2/5$	$1/5$	$\uparrow p_Y$

2. An urn contains 5 balls: two balls labeled 1, two balls labeled 2, and one ball labeled 3. You sample 2 balls from the bin, without replacement; the number on the first one is X, and the number on the second one is Y.

(c) (5 points) Are X and Y independent? Justify your answer.

No!

$$\mathbb{P}(X=3, Y=3) = 0$$

But $\mathbb{P}(X=3)\,\mathbb{P}(Y=3) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$

$$\mathbb{E}(U_j) = 0 = \frac{2+(-2)}{2}$$

3. Let $U_1, U_2, \ldots, U_n, \ldots$ be independent, identically distributed random variables, each with the $\mathrm{Uniform}[-2,2]$ distribution. Let $S_n = U_1 + U_2 + \cdots + U_n$.

$$f_{U_1}(x) = \frac{1}{4}\mathbb{1}_{[-2,2]}(x)$$

(a) (5 points) Compute $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$.

$$\mathbb{E}(S_n) = \mathbb{E}(U_1 + U_2 + \cdots + U_n) = \mathbb{E}(U_1) + \mathbb{E}(U_2) + \cdots + \mathbb{E}(U_n) = 0$$

$$\mathrm{Var}(S_n) = \mathrm{Var}(U_1 + U_2 + \cdots + U_n)$$

$$\overset{\text{indepent}}{=} \mathrm{Var}(U_1) + \mathrm{Var}(U_2) \cdots + \mathrm{Var}(U_n) = \frac{4}{3}n.$$

$$\mathrm{Var}(U_1) = \mathbb{E}(U_1^2) - \cancel{\underset{0}{\mathbb{E}(U_1)^2}}$$

$$\int_{-2}^{2} x^2 \frac{1}{4}\, dx = \frac{1}{3}x^3\frac{1}{4}\Big|_{-2}^{2} = \frac{2}{4}\cdot\frac{2^3}{3} = \frac{16}{3\cdot 4} = \frac{4}{3}.$$

3. Let $U_1, U_2, \ldots, U_n, \ldots$ be independent, identically distributed random variables, each with the $\mathrm{Uniform}[-2,2]$ distribution. Let $S_n = U_1 + U_2 + \cdots + U_n$.

(b) (5 points) For any $\epsilon > 0$, what can you say about

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{|S_n|}{n^{2/3}} \geq \epsilon\right)?$$

0

Chebyshev: $\mathbb{P}(|X - E(X)| \geq k\sigma(X)) \leq \frac{1}{k^2}$

$\mathbb{P}(|S_n - \underbrace{E(S_n)}_{=0}| \geq k\,\sigma(S_n)) \leq \frac{1}{k^2}$

$\sigma(S_n) = \sqrt{Var(S_n)} = \sqrt{\frac{4}{3}n}$

So $\mathbb{P}(|S_n| \geq k\sqrt{\frac{4}{3}n}) \leq \frac{1}{k^2}$.

$\therefore \mathbb{P}\left(\frac{|S_n|}{n^{2/3}} \geq \underbrace{k\frac{\sqrt{\frac{4}{3}n}}{n^{2/3}}}_{\varepsilon}\right) \leq \frac{1}{k^2}$

$\therefore \frac{k\sqrt{\frac{4}{3}n}}{n^{2/3}} = \varepsilon$

$k = \frac{\varepsilon n^{2/3}}{\sqrt{\frac{4}{3}n}}$

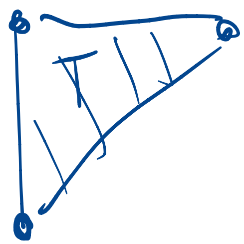
$\therefore \mathbb{P}\left(\frac{|S_n|}{n^{2/3}} \geq \varepsilon\right) \leq \frac{1}{\left(\frac{\varepsilon n^{2/3}}{\sqrt{\frac{4}{3}n}}\right)^2}$

$= \left(\frac{\sqrt{\frac{4}{3}n}}{\varepsilon n^{2/3}}\right)^2 = \frac{\frac{4}{3}n}{\varepsilon^2 n^{4/3}} = \frac{\frac{4}{3}/\varepsilon^2}{n^{1/3}}$

$\therefore \mathbb{P}\left(\frac{|S_n|}{n^{2/3}} \geq \varepsilon\right) \leq \frac{\text{const.}}{n^{1/3}} \to 0$ as $n \to \infty$.

4. Let T be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, and $(1,1)$ (including the interior). Suppose that $P=(X,Y)$ is a point chosen uniformly at random inside of T.

(a) (5 points) What is the joint density function of (X,Y)? Use this to compute $\mathrm{Cov}(X,Y)$.



(X,Y) uniform in T

$$f_{(X,Y)}(x,y) = \frac{1}{\text{Area}(T)} \mathbb{1}_T(x,y)$$

$$= 2\,\mathbb{1}_T(x,y)$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}.$$

$$\mathbb{E}(XY) = \iint_T xy\, f_{(X,Y)}(x,y)\,dx\,dy = 2\iint_T xy\,dx\,dy = \frac{1}{4}$$

$$\mathbb{E}(X) = \iint_T x\cdot 2\,dx\,dy = \frac{1}{3} \qquad \mathbb{E}(Y) = \iint_T y\cdot 2\,dx\,dy = 2/3$$

4. Let T be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, and $(1,1)$ (including the interior). Suppose that $P = (X,Y)$ is a point chosen uniformly at random inside of T.

(b) (5 points) Determine if X and Y are independent.

No! They are correlater:

$$\mathrm{Cov}(X,Y) = \frac{1}{36} \neq 0.$$

5. Suppose $X_1, X_2, \dots, X_n, \dots$ are i.i.d. random variables with mean $\mathbb{E}(X_j) = 0$ and variance $\mathrm{Var}(X_j) = 1$. Determine the following limits with precise justifications.

(a) (5 points) $\lim\limits_{n\to\infty} \mathbb{P}\left(-\frac{n}{4} \le X_1 + \cdots + X_n < \frac{n}{2}\right)$

5. Suppose $X_1, X_2, \ldots, X_n, \ldots$ are i.i.d. random variables with mean $\mathbb{E}(X_j) = 0$ and variance $\mathrm{Var}(X_j) = 1$. Determine the following limits with precise justifications.

(b) (5 points) $\lim_{n\to\infty} \mathbb{P}(X_1 + \cdots + X_n = 0)$

Please fill out your CAPEs !

And...

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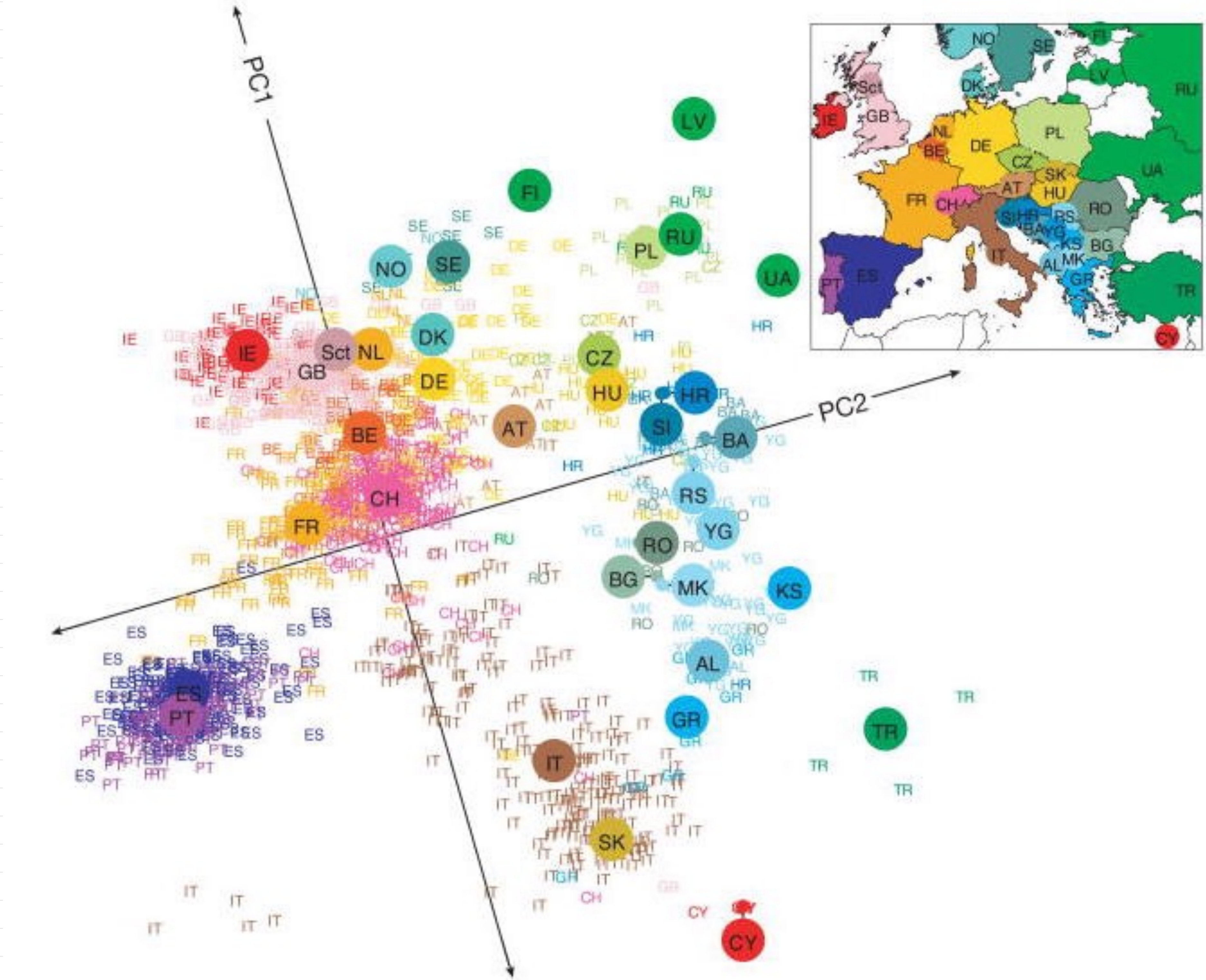
MATH 182 / DSC 155

Hidden Data in Random Matrices



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