MATH 180A: INTRO TO PROBABILITY

(FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

Today: § 9.2 - 9.3

Next: REVIEW

Lab 7 Due Wednesday, Dec 4.

Homework 8 Due Friday, Dec 6.

Final Exam: Monday, Dec 9, 11:30a–2:30p
in REC GYM

Reminder: Chebyshev's Inequality

9.1

For any random variable X with finite

$$\mathbb{E}(X) = \mu \qquad Var(X) = \sigma^2$$

$$\boxed{\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}}$$

We proved this using the fact that $\mathbb{E}$ is monotone:

$$X \leq Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$$

Eg. Ramen Menya Ultra has, on average, 1000 customers/day, with a standard deviation of 15. Estimate the probability that today they will have between 956 and 1044 customers.

(Weak) Law of Large Numbers

9.2

Let $X_1, X_2, X_3, \ldots, X_n, \ldots$ be an infinite sequence of i.i.d. random variables, each with $\mathbb{E}(X_j) = \mu$ and $\mathrm{Var}(X_j) = \sigma^2$ finite.

Let $S_n = X_1 + X_2 + \cdots + X_n$. Then for any fixed $\varepsilon > 0$, E.g. The Large Hadron Collider was built to detect and measure the mass of the Higgs Boson. Call the mass M.

For theoretical reasons, it is known that $M \leq 1.78 \times 10^{-23}\,g$.

How many trials do the LHC physicists need to do to estimate the correct mass (via sample mean) within $10^{-24}\,g$, with probability $\geq 95\%$?

Strong Law of Large Numbers

Let $X_1, X_2, X_3, \ldots, X_n, \ldots$ be an infinite sequence of i.i.d. random variables each with $\mathbb{E}(X_j) = \mu$.

Let $\bar{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$. Then

9.3

The Law of Large Numbers says that if $X_1, X_2, X_3, \dots$ are iid. with mean μ, and $S_n = X_1 + \dots + X_n$, then

$$\frac{S_n}{n} \to \mu$$

Central Limit Theorem

Let $X_1, X_2, \dots, X_n, \dots$ be iid random variables with $E(X_j) = \mu$, $Var(X_j) = \sigma^2$.

Then , and for $-\infty < a \leq b < \infty$, Proof. Let $Y_n = (S_n - n\mu)/\sigma\sqrt{n}$. We will show that

$$M_{Y_n}(t) \longrightarrow M_{\mathcal{N}(0,1)}(t) = e^{t^2/2} \text{ for all } t \in \mathbb{R}.$$

Rate of Convergence in CLT

Theorem (Berry – Esseen, 1941-42)

$$\left| \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x \right) - \Phi(x) \right| \le \frac{3\,\mathbb{E}\left[|X_1-\mu|^3\right]}{\sigma^3\sqrt{n}}$$

Eg. What does the p.d.f. of a Poisson(n) r.v. look like for large n?