MATH 180A: INTRO TO PROBABILITY
(FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

Today: § 6.3, 8.1

Next: § 8.2 - 8.4

Lab 6 Due TONIGHT

Homework 7 Due Wednesday, Nov 27

Joint Distributions & Independence

6.3

Suppose $\underline{X} = (X_1, \ldots, X_n)$ is jointly continuous. Then $X_1, \ldots, X_n$ are independent iff $f_{\underline{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$.

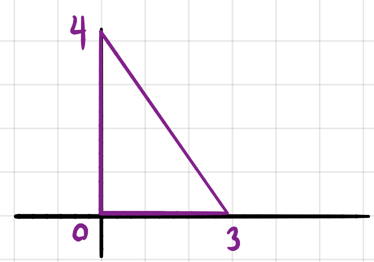
$(\Longrightarrow)$

Eg. Multivariate standard normal $\frac{1}{(2\pi)^{n/2}} e^{-\|\underline{x}\|^2/2}$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}\right) \cdots \left(\frac{1}{\sqrt{2\pi}} e^{-x_n^2/2}\right)$$

Since the joint density is a product of single-variable densities, the components of the random vector are independent, with those densities as marginals:

Q: S'pose $\underset{\sim}{X}$ is uniform on the triangle $\underset{\sim}{X} = (X, Y)$. Are X, Y independent?



Example. Suppose $X \sim \mathrm{Exp}(\lambda)$, $Y \sim \mathrm{Exp}(\mu)$, and X, Y independent. Find the distribution of $\min(X, Y)$.

Theorem. Suppose $\underline{X} = (X_1, \dots, X_n)$ is a random vector, with $X_1, \dots, X_n$ independent.

For any functions $g_1, g_2, \dots, g_n : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}(g_1(X_1)\, g_2(X_2) \cdots g_n(X_n)) = \mathbb{E}(g_1(X_1)) \cdot \mathbb{E}(g_2(X_2)) \cdots \mathbb{E}(g_n(X_n))$$

In particular, $\mathbb{E}(X_1 X_2 \cdots X_n) = \mathbb{E}(X_1)\mathbb{E}(X_2) \cdots \mathbb{E}(X_n)$.

Actually, the last theorem is iff.

↳ Suppose we know

$$\mathbb{E}\left(g_1(X_1)g_2(X_2)\cdots g_n(X_n)\right) = \mathbb{E}(g_1(X_1))\,\mathbb{E}(g_2(X_2))\cdots\mathbb{E}(g_n(X_n))$$

for all functions $g_1, g_2, \dots, g_n$.

Convolution

Let X, Y be independent random variables. What can we say about the distribution of $X+Y$?

The convolution of two prob. densities is $(f*g)(t)$

Eg. $X, Y \sim \text{Exp}(\lambda)$, independent. Find f_{X+Y}.

E.g. $X, Y \sim \mathrm{Unif}([0,1])$, independent.

8.1

We can now come back to questions about sums of random variables – in the context of their joint distribution.

Let X, Y be two (let's say discrete) random variables.

$\mathbb{E}(X+Y)$

Theorem: For any random variables $X_1, X_2, \dots, X_n$,

$\mathbb{E}(X_1 + \dots + X_n)$

Eg. $S \sim \text{Bin}(n,p)$.

This means $S = X_1 + X_2 + \cdots + X_n$ where $X_1, \ldots, X_n \sim \text{Ber}(p)$

A binomial is a sum of Bernoullis (indicator r.v.'s).
Lots of problems can be solved when we can express desired events in terms of sums of indicators.

Eg. Suppose we put 200 balls randomly into 100 boxes. What is the expected number of empty boxes?

Eg. Your favorite cereal (chocolate frosted sugar bombs) comes with a Pokémon figurine. There are 20 to collect. What is the expected number of boxes you need to buy to collect them all?