# MATH 180A: INTRO TO PROBABILITY
## (FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

**Midterm Exam TONIGHT!**

8pm CENTR 101

* Assigned seats (see TritonEd)
* Bring Student ID
* 1 Double-Sided 8.5"x11" sheet of hand-written notes
* no electronic devices
* eat a proper dinner before (but not right before)
* try to relax; have fun!

Name: ____________________ PID: ____________________ **(P. 3)**

**1**. Suppose that the time it takes for you to complete your probability homework is distributed according to an exponential random variable with mean 1 hour. You start your homework at 8:00 PM. Your bedtime is 10:00 PM. If you finish your homework before your bedtime, you watch TV until your bedtime and then go to sleep. If you do not finish by your bedtime, you go to sleep anyway, and so you do not watch TV at all. Let $Y$ be the random variable that measures the amount of time in hours that you spend watching TV.

**(a)** (6 points) Calculate the CDF of $Y$.

$X$ = time to do HW $\sim \mathrm{Exp}(1)$

$f_X(x) = e^{-x} 1_{x \geq 0}$

$$Y = \begin{cases} 2 - X & \text{if } X < 2 \\ 0 & \text{if } X \geq 2 \end{cases} = g(X)$$

$Y \in [0,2]$

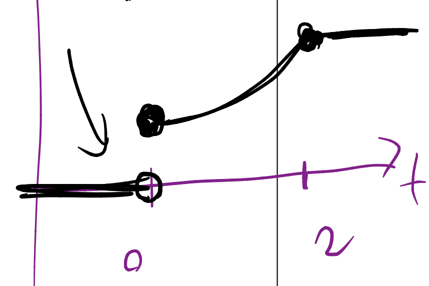
$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 2 \end{cases}$$

$t \in [0,2]$

$$P(Y \leq t) = P(2 - X \leq t) = P(X \geq 2 - t) = \int_{2-t}^{\infty} e^{-x}\,dx = -e^{-x}\Big|_{2-t}^{\infty} = e^{-(2-t)}$$

**(b)** (4 points) Calculate the expected value $\mathbb{E}[Y]$.

$P(Y=0) = e^{-2} \neq 0.$



$$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = \int_0^{\infty} g(x) e^{-x}\,dx = \int_0^2 (2-x) e^{-x}\,dx$$

(int. by parts)

**6.** [**Extra**] (10 points) Show that there is no random variable $X$ with moment generating function $M_X(t)$ such that $M_X(1) = 3$ and $M_X(2) = 4$.

$$\mathrm{Var}(Y) = \mathbb{E}((Y - \mathbb{E}(Y))^2) \geq 0$$

$$= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$$

$$M_X(t) = \mathbb{E}(e^{tX})$$

$$\left.\begin{aligned} \mathbb{E}(Y) &= \mathbb{E}(e^{1X}) = 3 \\ \mathbb{E}(Y^2) &= \mathbb{E}(e^{2X}) = 4 \\ &\text{Moments of } Y = e^X. \end{aligned}\right\} \quad \mathrm{Var}(Y) = 4 - 3^2 = -5$$

!

**4.** Let $X$ be a continuous random variable with m.g.f. $M_X(t) = e^{8t^2}$.

**(a)** (4 points) Find the p.d.f. of $X$.

If $Z \sim N(0,1)$, $M_Z(t) = e^{t^2/2}$

$\therefore M_{aZ}(t) = \mathbb{E}(e^{taZ}) = \mathbb{E}(e^{(at)Z}) = M_Z(at) = e^{\frac{(at)^2}{2}} = e^{\frac{a^2}{2}t^2}$

need $\frac{a^2}{2} = 8$, $a = 4$

So $M_X$ is the mgf of $4Z$.

$\therefore$ by uniqueness of MGF $\Leftrightarrow$ distrib.

(b/c MGF $< \infty$ on $(-\varepsilon, \varepsilon)$),

$\Rightarrow f_X(t) = f_{4Z}(t)$

$F_X(t) = F_{4Z}(t) = \mathbb{P}(4Z \le t) = \mathbb{P}(Z \le t/4) = \Phi(t/4)$

$\therefore f_X(t) = \frac{d}{dt}\mathbb{P}(Z \le \frac{t}{4}) = \Phi'(\frac{t}{4})\frac{1}{4} = \frac{1}{4\sqrt{2\pi}} e^{-(t/4)^2/2}$

$Y \sim N(\mu, \sigma^2)$

$M_Y(t) = e^{\frac{\sigma^2}{2}t^2 + \mu t} = e^{8t^2}$

$\mu = 0$

$\therefore \frac{\sigma^2}{2} = 8$

**(b)** (6 points) Compute the p.d.f. of the random variable $X^3 + 1$ in terms of the p.d.f. of $X$.

$F_{X^3+1}(t) = \mathbb{P}(X^3 + 1 \le t) = \mathbb{P}(X^3 \le t-1) = \mathbb{P}(X \le (t-1)^{1/3}) = F_X((t-1)^{1/3})$.

$\therefore f_{X^3+1}(t) = \frac{d}{dt}\left(F_X((t-1)^{1/3})\right) = F_X'((t-1)^{1/3}) \cdot \frac{1}{3}(t-1)^{-2/3}$

$= \frac{1}{12}\frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{(t-1)^{2/3}}{32}}}{(t-1)^{2/3}}$.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

To illustrate the use of the table: $\Phi(0.36) = 0.6406$, $\Phi(1.34) = 0.9099$

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7703 | 0.7734 | 0.7764 | 0.7793 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8943 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9924 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9986 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |

$X \sim \text{Poisson}(\lambda)$

Find $p_{e^X}$.

Possible values of $e^X$:

$X \in \{0, 1, 2, 3, \dots\} \in \mathbb{N}.$

$e^X \in \{1, e, e^2, e^3, \dots\} \leftarrow t$

$$p_{e^X}(t) = \mathbb{P}(e^X = t)$$

$$= \underbrace{\mathbb{P}(X = \ln t)}_{\neq 0 \text{ iff } \ln t \in \mathbb{N}} = e^{-\lambda} \frac{\lambda^{\ln t}}{(\ln t)!}$$

Eg. $Y = \sin\left(\frac{\pi}{2} X\right) \in \{-1, 0, 1\}$

$$p_Y(0) = \mathbb{P}(Y=0) = \mathbb{P}\left(\tfrac{\pi}{2} X \in \{0, \pi, 2\pi, 3\pi, \dots\}\right) = \mathbb{P}(X \in 2\mathbb{N}) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{2k}}{(2k)!}$$

$$p_Y(1) = \mathbb{P}(Y=1) = \mathbb{P}\Big(\underbrace{\tfrac{\pi}{2} X \in \left\{\tfrac{\pi}{2}, \tfrac{5\pi}{2}, \tfrac{9\pi}{2}, \dots\right\}}_{X \in \{1, 5, 9, 13, \dots\}}\Big) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{4k+1}}{(4k+1)!}$$