MATH 180A: INTRO TO PROBABILITY
(FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

Today: § 6.2-6.3

Next: § 8.1-8.3

Midterm 2: Wednesday 8pm ← CENTR 101 !

- covering Chapters 3-5. ~~Poisson Process~~
- seat assignment on TritonEd
- bring student ID
- bring 1 sheet (double-sided) hand-written notes.
- bring your A GAME!

Jointly Continuous Random Vectors

A random vector $\underline{X} = (X_1, \dots, X_n)$ has a pdf $f_{\underline{X}} : \mathbb{R}^n \to \mathbb{R}_+$ if, for "nice" subsets $B \subseteq \mathbb{R}^n$

$$\mathbb{P}(\underline{X} \in B) = \int_B f_{\underline{X}}(x_1, \dots, x_n)\, dx_1 \cdots dx_n .$$

(we say $X_1, X_2, \dots, X_n$ are jointly continuous.)

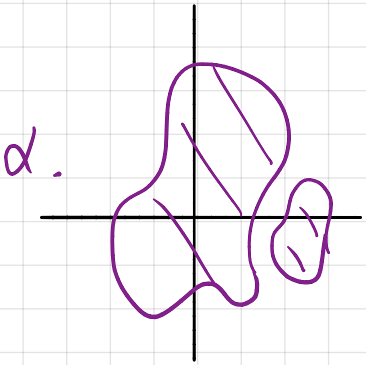
Properties: $f_X(\underline{x}) \geq 0$ for all $\underline{x} \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(\underline{x})\, d\underline{x} = 1 .$$

Eg. Standard Multivariate Normal $\quad f(\underline{x}) = (2\pi)^{-\frac{n}{2}} e^{-\|\underline{x}\|^2/2}$

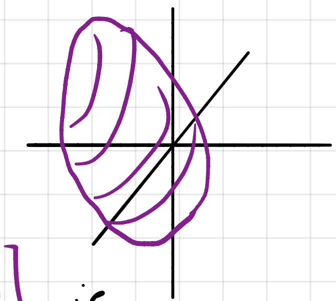
E.g. Uniform Probability

Let A be a bounded region in $\mathbb{R}^2$ with area α.



Let V be a bounded region in $\mathbb{R}^3$ with volume ϑ

$\vdots$



A random vector $\underline{X} = (X_1, X_2)$ [resp. $\underline{X} = (X_1, X_2, X_3)$] is **uniformly distributed in** A [resp. V] if it is jointly continuous and has density

$$f_A(x_1, x_2) = \frac{1}{\alpha}\mathbb{1}\big((x_1, x_2) \in A\big) \quad \text{resp.} \quad f_V(x_1, x_2, x_3) = \frac{1}{\vartheta}\mathbb{1}\big((x_1, x_2, x_3) \in V\big)$$

E.g. Suppose (X,Y) has joint density $f(x,y) = \frac{3}{2}(xy^2+y)\,\mathbb{1}(0 \le x,y \le 1)$

Compute $\mathbb{P}(X<Y)$.

Marginals

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a jointly continuous random vector, with joint density $f_{\underline{X}}$. The density of X_j is

$$f_{X_j}(t) = \int_{\mathbb{R}^{n-1}} f_{\underline{X}}(x_1, x_2, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n .$$

(Integrate out all but the j^{th} variable)

Eg. Suppose $\underline{X}$ is uniformly distributed on the disk of radius 2.
Find the marginal density of X_1 $\quad (\underline{X} = (X_1, X_2))$

Expectation

If $\underline{X}$ is a random vector in $\mathbb{R}^n$ with joint density $f_{\underline{X}} : \mathbb{R}^n \to \mathbb{R}_+$, and $g : \mathbb{R}^n \to \mathbb{R}$, then

$$\mathbb{E}(g(\underline{X})) =$$

Eg. $\underline{X} = (X, Y)$, $f_{\underline{X}}(x,y) = \frac{3}{2}(xy^2 + y)\, \mathbb{1}(0 \leq x, y \leq 1)$. Find $\mathbb{E}(X^2 Y)$.

CAUTION !

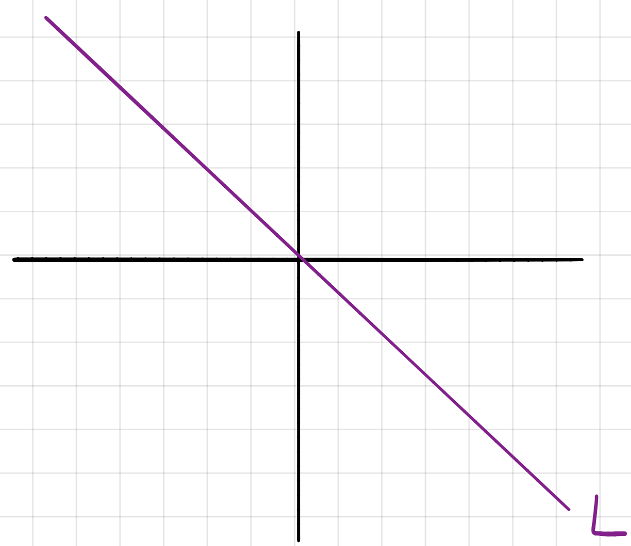
If $X_1, X_2, \dots, X_n$ are discrete random variables, then $\underline{X} = (X_1, \dots, X_n)$ is a (jointly) discrete random vector.

BUT

Just because $X_1, \dots, X_n$ are (separately) continuous random variables does not necessarily imply that $\underline{X} = (X_1, \dots, X_n)$ has a joint density!

E.g. $X \sim \mathcal{N}(0,1)$, $Y = -X$.

Must have $\mathbb{P}(X = -Y) = 1$.



Joint Distributions & Independence

6.3

Suppose $\underline{X} = (X_1, \ldots, X_n)$ is jointly continuous. Then $X_1, \ldots, X_n$ are independent iff

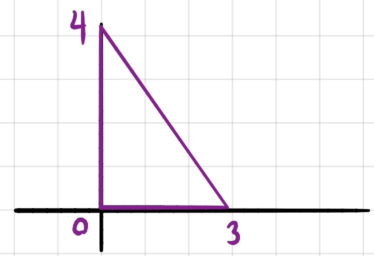
E.g. Multivariate standard normal $\frac{1}{(2\pi)^{n/2}} e^{-\|\underline{x}\|^2/2}$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}\right) \cdots \left(\frac{1}{\sqrt{2\pi}} e^{-x_n^2/2}\right)$$

Since the joint density is a product of single-variable densities, the components of the random vector are independent, with those densities as marginals:

E.g. S'pose $\underset{\sim}{X} = (X, Y)$ is uniform on the triangle

Are X, Y independent?



Theorem. Suppose $\underline{X} = (X_1, \ldots, X_n)$ is a random vector, with $X_1, \ldots, X_n$ independent.

For any functions $g_1, g_2, \ldots, g_n : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\left(g_1(X_1)\, g_2(X_2) \cdots g_n(X_n)\right) = \mathbb{E}(g_1(X_1)) \cdot \mathbb{E}(g_2(X_2)) \cdots \mathbb{E}(g_n(X_n))$$

In particular, $\mathbb{E}(X_1 X_2 \cdots X_n) = \mathbb{E}(X_1)\mathbb{E}(X_2) \cdots \mathbb{E}(X_n)$.