MATH 180A: INTRO TO PROBABILITY
(FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

Today: § 6.1 - 6.3

Next: § 8.1 - 8.3

HW 6 due TONIGHT by 11:59pm

Midterm 2: Next Wednesday ← CENTR 101 !
covering Chapters 3 - 5.

Definition: Given (discrete) random variables $X_1, X_2, \dots, X_n$ all defined on the same sample space, their **joint distribution** is the collection of all

$$\left\{ \mathbb{P}(X_1=k_1, X_2=k_2, \dots, X_n=k_n) \;\middle|\; \text{all possible values } k_1 \text{ of } X_1,\ k_2 \text{ of } X_2, \dots, k_n \text{ of } X_n \right\}$$

Eg. $X, Y \sim Ber(p)$,

		X	
		0	1
Y	0		
	1		

Recovering X_j from $\underline{X} = (X_1, X_2, \dots, X_n)$: <u>Marginals</u>

Suppose we know $p_{\underline{X}}(\underline{k})$ for all $\underline{k} = (k_1, k_2, \dots, k_n)$.

How can we find $p_{X_1}(t)$?

Eg. Toss a fair coin 3 times. X = # tails in first toss (0 or 1)
Y = total # tails in all 3 (0, 1, 2, 3)

		Y			
		0	1	2	3
X	0				
	1				

Question: Are X, Y independent?

Joint distributions are just distributions of random vectors.

$X_1, X_2, \dots, X_n \rightsquigarrow \underline{X} = (X_1, X_2, \dots, X_n)$

possible values for $\underline{X}$ are vectors $(k_1, \dots, k_n)$.

Eg. Multinomial Distribution.

Often, trials have more than 2 outcomes.

Consider a trial w/ r possible outcomes, w/ probabilities $p_1, p_2, \dots, p_r$

Perform n trials.

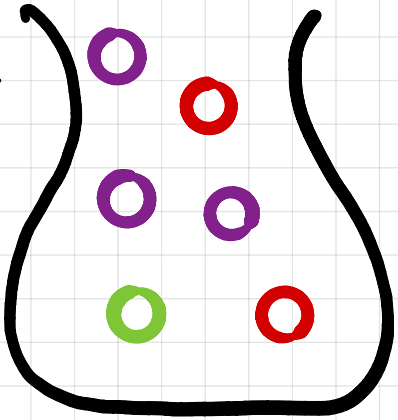
For $1 \le j \le r$, X_j = # times we get outcome j.

Possible values for $\underline{X} = (X_1, \dots, X_r)$:

$\mathbb{P}(\underline{X} = \underline{k})$

pmf of $\mathrm{Mult}(n; p_1, \dots, p_r)$

Eg.



Sample 10 times with replacement.

$\mathbb{P}$(3 green, 2 red, 5 blue)

Note: if $r=2$, $\text{Mult}(n; p, q)$

Eg. Suppose $\underline{X} \sim \text{Mult}(n; p_1, p_2, \ldots, p_r)$. Find the marginal distribution of X_1.

Expectations

Let $\underline{X} = (X_1, \dots, X_n)$ be a (discrete) random vector with joint probability mass function $p_{\underline{X}}$.

If $g: \mathbb{R}^n \to \mathbb{R}$ is a function, $Y = g(\underline{X})$ is a random variable (still discrete).

$$\mathbb{E}(Y) = \sum_t t \cdot \mathbb{P}(Y = t)$$

Eg. Toss a fair coin twice, $X_1, X_2 \in \{0, 1\}$.

$$\mathbb{E}(X_1 X_2)$$

Jointly Continuous Random Vectors

A random vector $\underline{X} = (X_1, \dots, X_n)$ has a pdf $f_{\underline{X}} : \mathbb{R}^n \to \mathbb{R}_+$ if, for "nice" subsets $B \subseteq \mathbb{R}^n$

$$\mathbb{P}(\underline{X} \in B) = \int_B f_{\underline{X}}(x_1, \dots, x_n)\, dx_1 \cdots dx_n .$$

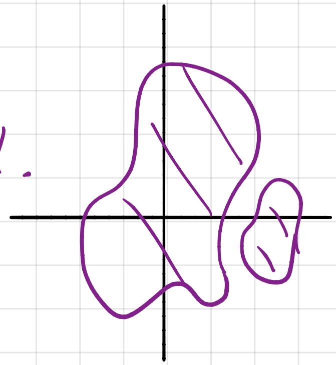
(we say $X_1, X_2, \dots, X_n$ are jointly continuous.)

Properties:

Eg. Standard Multivariate Normal $\quad f(x_1, \dots, x_n) = (2\pi)^{-n/2} e^{-\frac{(x_1^2 + \dots + x_n^2)}{2}}$

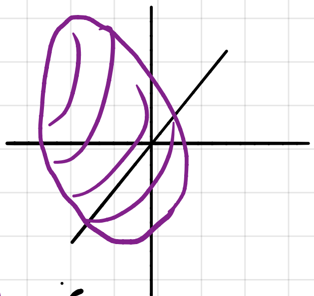
E.g. Uniform Probability

Let A be a bounded region in $\mathbb{R}^2$ with area α.



Let V be a bounded region in $\mathbb{R}^3$ with volume ϑ

$\vdots$



A random vector $\underline{X} = (X_1, X_2)$ [resp. $\underline{X} = (X_1, X_2, X_3)$] is **uniformly distributed in** A [resp. V] if it is jointly continuous and has density

$$f_A(x_1, x_2) = \frac{1}{\alpha} \mathbb{1}\big((x_1, x_2) \in A\big) \quad \text{resp.} \quad f_V(x_1, x_2, x_3) = \frac{1}{\vartheta} \mathbb{1}\big((x_1, x_2, x_3) \in V\big)$$

Eg. Suppose (X, Y) has joint density $f(x,y) = \frac{3}{2}(xy^2+y)\,\mathbb{1}(0 \leq x,y \leq 1)$

Compute $\mathbb{P}(X < Y)$.

Marginals

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a jointly continuous random vector, with joint density $f_{\underline{X}}$. The density of X_j is

$$f_{X_j}(t) = \int_{\mathbb{R}^{n-1}} f_{\underline{X}}(x_1, x_2, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n .$$

(Integrate out all but the j^{th} variable)

E.g. Suppose $\underline{X}$ is uniformly distributed on the disk of radius 2.
Find the marginal density of X_1 $\quad (\underline{X} = (X_1, X_2))$