MATH 180A: INTRO TO PROBABILITY
(FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

Today: §6.1–6.2

Next: §6.3, 8.1–8.3

HW 6 due TONIGHT by 11:59pm

Lab 6 due Friday, Nov 21, by 11:59 pm

Midterm 2: Next Wednesday ← CENTR 101!

covering Chapters 3-5

practice midterms posted this weekend.

Definition: Given (discrete) random variables $X_1, X_2, \dots, X_n$ all defined on the same sample space, their **joint distribution** is the collection of all

$$\left\{ \mathbb{P}(X_1 = k_1, X_2 = k_2, \dots, X_n = k_n) \;\; \text{all possible values } k_1 \text{ of } X_1,\ k_2 \text{ of } X_2, \dots, k_n \text{ of } X_n \right\}$$

Eg. $X, Y \sim \mathrm{Ber}(p)$,

(1) $X = Y$

(2) X, Y independent

$$\mathbb{P}(X = k_1, Y = k_2) = \mathbb{P}(X = k_1)\mathbb{P}(Y = k_2)$$

$Y \backslash X$	0	1
0	$1-p$ $\quad (1-p)^2$	0 $\quad p(1-p)$
1	0 $\quad (1-p)p$	p $\quad p^2$

$$p_{X,Y}(k_1, k_2) = \mathbb{P}(X = k_1, Y = k_2)$$

Recovering X_j from $\underline{X} = (X_1, X_2, \dots, X_n)$: Marginals

Suppose we know $p_{\underline{X}}(\underline{k})$ for all $\underline{k} = (k_1, k_2, \dots, k_n)$.
How can we find $p_{X_1}(t)$?

E.g. Toss a fair coin twice. X_1, X_2. $\in \{0,1\}$

$$\underbrace{\qquad}_{} \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$$\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 0, X_2 = 0) + \mathbb{P}(X_1 = 0, X_2 = 1)$$

$$\{X_1 = 0\} = \{X_1 = 0, X_2 = 0\} \cup \{X_1 = 0, X_2 = 1\}$$

In general,

$$\mathbb{P}(X_1 = t) = \sum_{k_2, k_3, \dots, k_n} \mathbb{P}(X_1 = t, X_2 = k_2, \dots, X_n = k_n) = \sum_{k_2, \dots, k_n} p_{\underline{X}}(t, k_2, k_3, \dots, k_n)$$

$$\{X_1 = t\} = \bigcup_{k_2, \dots, k_n} \{X_1 = t, X_2 = k_2, X_3 = k_3, \dots, X_n = k_n\}$$

$$\mathbb{P}(X = t) = \sum_k p_{(X,Y)}(t, k) \qquad \mathbb{P}(Y = t) = \sum_k p_{(X,Y)}(k, t)$$

(X, Y)

Eg. Toss a fair coin 3 times. X = # tails in first toss (0 or 1)

Y = total # tails in all 3 (0,1,2,3)

$X \backslash Y$	0	1	2	3	$\mathrm{Ber}(\frac{1}{2})$
0	1/8	1/4	1/8	0	→ 1/2
1	0	1/8	1/4	1/8	→ 1/2
$\mathrm{Bin}(3,\frac{1}{2})$	1/8	3/8	3/8	1/8	

Outcome	X	Y
HHH	0	0
HHT	0	1
HTH	0	1
HTT	0	2
THH	1	1
THT	1	2
TTH	1	2
TTT	1	3

each has $p = \frac{1}{8}$.

$\mathbb{P}(X=k, Y=l)$

k	l
0	0
1	1
	2
	3

Question: Are X, Y independent?

$\mathbb{P}(X=1, Y=0) = 0$

$\mathbb{P}(X=1)\mathbb{P}(Y=0) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16} \neq 0.$

No!

Joint distributions are just distributions of random vectors.

$X_1, X_2, \dots, X_n \rightsquigarrow \underline{X} = (X_1, X_2, \dots, X_n)$

$\overbrace{\text{possible}}$ values for $\underline{X}$ are vectors $(k_1, \dots, k_n)$.

Eg. Multinomial Distribution.

Often, trials have more than 2 outcomes.

Consider a trial w/ r possible outcomes, w/ probabilities

$p_1, p_2, \dots, p_r$

$p_1 + p_2 + \dots + p_r = 1$

Perform n trials.

For $1 \le j \le r$, X_j = # times we get outcome j.

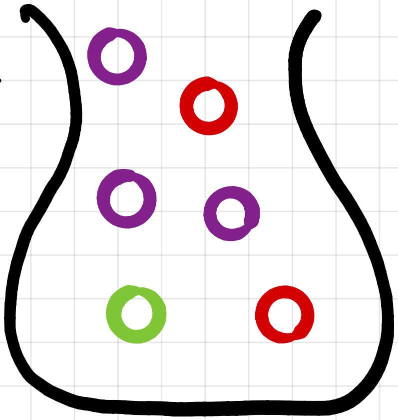
Possible values for $\underline{X} = (X_1, \dots, X_r)$: $(k_1, \dots, k_r) \leftarrow k_j \in \{0, 1, \dots, n\}$, $k_1 + \dots + k_r = n$

$$\mathbb{P}(\underline{X} = \underline{k}) = \begin{pmatrix} \text{\# arrangements} \\ \text{w/ } k_1 \text{ 1s, } k_2 \text{ 2s}, \dots, k_r \text{ r's} \end{pmatrix} p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

$(r=2)$: $\left(\dfrac{n!}{k_1! \, k_2!} = \dfrac{n!}{k_1!(n-k_1)!} = \dbinom{n}{k_1} \right)$

pmf of $\mathrm{Mult}(n; p_1, \dots, p_r)$

$$\binom{n}{k_1, k_2, \dots, k_r} = \frac{n!}{k_1! \, k_2! \cdots k_r!}$$

Eg. 

Sample 10 times with replacement.

$\mathbb{P}(3 \text{ green}, 2 \text{ red}, 5 \text{ blue})$

$G = \#\text{green}, \quad R = \#\text{reds}, \quad B = \#\text{blues}$

$$\underline{X} = (G, R, B) \sim \text{Mult}\left(10; \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}\right)$$

$$\mathbb{P}(\underline{X} = (3, 2, 5)) = \frac{10!}{3!\,2!\,5!} \left(\frac{1}{6}\right)^3 \left(\frac{1}{3}\right)^2 \left(\frac{1}{2}\right)^5 \doteq 4.05\%$$

Note: if $r = 2$, $\text{Mult}(n; p, q)$

Eg. Suppose $\underline{X} \sim \text{Mult}(n; p_1, p_2, \dots, p_r)$. Find the marginal distribution of X_1.

$$\mathbb{P}(X_1 = t) = \sum_{\substack{k_2, k_3, \dots, k_r \geq 0 \\ t + k_2 + \dots + k_r = n}} \frac{n!}{t!\, k_2! \cdots k_r!} p_1^t p_2^{k_2} \cdots p_r^{k_r}$$

Complicated! Instead observe:

$X_1 = \#\text{successes in } n \text{ indep. trials}$

where success = outcome 1 $(\mathbb{P} = p_1)$

failure = outcomes 2–r $(\mathbb{P} = p_2 + \dots + p_r = 1 - p)$

$$X_1 \sim \text{Bin}(n, p)$$

Expectations

Let $\underline{X} = (X_1, \dots, X_n)$ be a (discrete) random vector with joint probability mass function $p_{\underline{X}}$.

If $g : \mathbb{R}^n \to \mathbb{R}$ is a function, $Y = g(\underline{X})$ is a random variable (still discrete).

$$\mathbb{E}(Y) = \sum_t t \cdot \mathbb{P}(Y = t)$$

$$= \sum_{\underline{k}} g(\underline{k})\, \mathbb{P}(\underline{X} = \underline{k})$$

$$= \sum_{\underline{k}} g(\underline{k})\, p_{\underline{X}}(\underline{k})$$

Eg. Toss a fair coin twice, $X_1, X_2 \in \{0, 1\}$.

$$\mathbb{E}(X_1 X_2) = \sum_{k_1, k_2} \underbrace{g(k_1, k_2)}_{k_1 k_2}\, \underbrace{\mathbb{P}(X_1 = k_1, X_2 = k_2)}_{1/4}$$

$$= \cancel{0 \cdot 0 \cdot \tfrac{1}{4}} + \cancel{0 \cdot 1 \cdot \tfrac{1}{4}} + \cancel{1 \cdot 0 \cdot \tfrac{1}{4}} + 1 \cdot 1 \cdot \tfrac{1}{4}$$

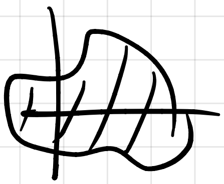
$$= 1/4 \leftarrow \mathbb{E}(X)\mathbb{E}(Y)$$

$\mathbb{E}(X) = 1/2$

$\mathbb{E}(Y) = 1/2$

Jointly Continuous Random Vectors

A random vector $\underline{X} = (X_1, \dots, X_n)$ has a pdf $f_{\underline{X}} : \mathbb{R}^n \to \mathbb{R}_+$ if, for "nice" subsets $B \subseteq \mathbb{R}^n$



$$\mathbb{P}(\underline{X} \in B) = \int_B f_{\underline{X}}(x_1, \dots, x_n)\, dx_1 \cdots dx_n .$$

(we say $X_1, X_2, \dots, X_n$ are jointly continuous.)

Properties: (1) $f_{\underline{X}} \geq 0$

(2) $\int_{\mathbb{R}^n} f_{\underline{X}} = 1$.

Eg. Standard Multivariate Normal $\quad f(x_1, \dots, x_n) = (2\pi)^{-n/2} \underbrace{e^{-\frac{(x_1^2 + \dots + x_n^2)}{2}}}_{e^{-x_1^2/2}\, e^{-x_2^2/2} \cdots e^{-x_n^2/2}}$

$$\underbrace{\int_{\mathbb{R}^n} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n}_{1} = (2\pi)^{-n/2} \int_{-\infty}^{\infty} e^{-x_1^2/2}\, dx_1 \cdot \int_{-\infty}^{\infty} e^{-x_2^2/2}\, dx_2 \cdots \int_{-\infty}^{\infty} e^{-x_n^2/2}\, dx_n = (2\pi)^{-n/2} (\sqrt{2\pi})(\sqrt{2\pi}) \cdots (\sqrt{2\pi})$$