MATH 180A: INTRO TO PROBABILITY
(FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

Today: §3.5, 4.1

Next: §4.2-4.3

HW3 grades released; regrade requests Tue 10/29 8am-11pm

Midterm grades released; regrade requests Wed 10/30 8am-11pm

Lab 3 grades released; regrade requests Thu 10/31 8am-11pm

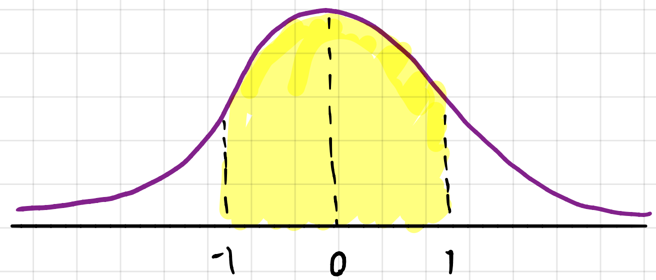
HW4 now posted; due next Friday (Nov 1) by 11:59pm.

Standard Normal / Gaussian $\mathcal{N}(0,1)$

Probability density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Eg. Let $X \sim \mathcal{N}(0,1)$. What is $\mathbb{P}(|X| \leq 1)$?



$$\mathbb{P}(|X| \leq 1) =$$

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

Normal Table of Values

This standard table (Appendix E in textbook) lists the values of

Eg. $\Phi(1.56)$

Fact: $\Phi(-x) = 1 - \Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Mean and Variance $\quad X \sim \mathcal{N}(0,1)$

$\mathbb{E}(X) =$

$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2]$

General Normal $\mathcal{N}(\mu, \sigma^2)$

Let $X \sim \mathcal{N}(0,1)$. For $\sigma > 0$, $\mu \in \mathbb{R}$, let $Y = \sigma X + \mu$.

$$\mathbb{P}(Y \leq y) = \mathbb{P}(\sigma X + \mu \leq y)$$

$$\therefore \quad f_Y(y) = \frac{d}{dy}\mathbb{P}(Y \leq y)$$

Fact: If $a, b \in \mathbb{R}$, $\mathbb{E}(aX+b) = a\mathbb{E}(X)+b$ & $Var(aX+b) = a^2 Var(X)$

$\mathbb{E}(\sigma X+\mu) = \sigma\mathbb{E}(X)+\mu = 0+\mu = \mu$. $Var(\sigma X+\mu) = \sigma^2 Var(X) = \sigma^2$

$$\therefore \quad \mathbb{P}(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2} \quad \text{(Chebyshev)}$$

Why should I care about normal distributions?

4.1

We've already seen one scaling limit: if $S_n \sim Bin(n,p)$
$p = \lambda/n$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(S_n = k) =$$

This is for rare events.

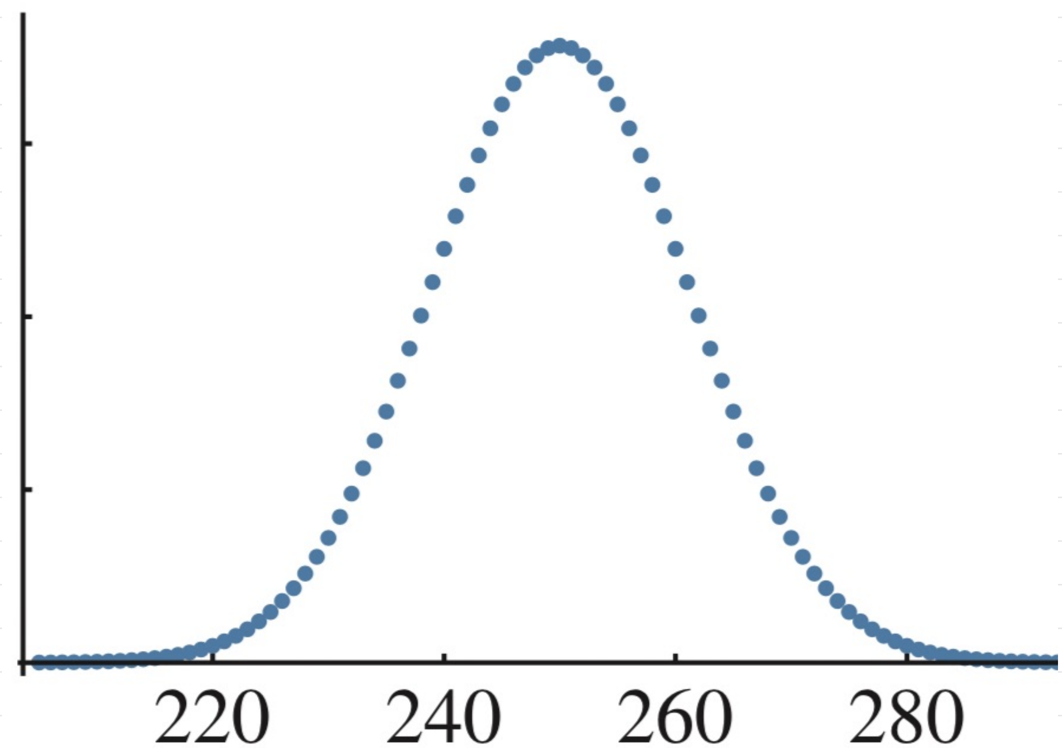
But what if we are sampling trials where success is not so rare?

Eg. Toss a fair coin 500 times. What is the probability that the number of heads is between 240 and 260?

Here is a plot of the probability mass function of the $\mathrm{Bin}(500, \frac{1}{2})$ distribution.

It has a very distinct bell curve shape.

This is no accident: as $n \to \infty$, for fixed p, $\mathrm{Bin}(n,p)$ approximates a normal distribution!



Which one? Determined by mean and variance.

$S_n \sim \mathrm{Bin}(n,p)$

↳ $\mathbb{E}(S_n) =$

↳ $\mathrm{Var}(S_n) =$

Vague Theorem

For n large and p not close to 0 or 1,

$$\mathrm{Bin}(n,p) \approx$$

Binomial Central Limit Theorem

Fix $p \in (0,1)$. For each n, let $S_n \sim \text{Bin}(n,p)$.
For any fixed $a \le b$,

$$\lim_{n\to\infty} \mathbb{P}\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b \right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx .$$

E.g. Toss a fair coin 500 times. What is the probability that the number of heads is between 240 and 260?

Eg. A fair die is rolled 720 times. What is the probability that exactly 113 sixes come up?

$S = \#\text{sixes} \sim \text{Bin}(720, \frac{1}{6})$. $\mathbb{P}(S=113)$

Normal Approximation:

$\mathbb{E}(S) = 720 \cdot \frac{1}{6} = 120$ $\quad$ $\text{Var}(S) = 720 \cdot \frac{1}{6} \cdot \frac{5}{6} = 100$

$\mathbb{P}(S=113)$

Fun Example: Roll a fair die. If it comes up 1 or 2, take 2 steps.
If it comes up 3 or more, take 3 steps.

Repeat. After 450 rolls, how likely is it you've taken more than 1185 steps?