# MATH 180A: INTRO TO PROBABILITY
## (FOR DATA SCIENCE)

www.math.ucsd.edu/~tkemp/180A

## Midterm Exam TONIGHT!

8pm PCYNH 109

* Assigned seats (see TritonEd)
* Bring Student ID
* 1 Double-Sided 8.5"x11" sheet of hand-written notes
* no electronic devices
* eat a proper dinner before (but not right before)
* try to relax; have fun!

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Regrade requests for Lab 2: Thursday, 10/24 8am – 11pm

**1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Suppose that $A, B \in \mathcal{F}$ satisfy

$$\mathbb{P}(A) + \mathbb{P}(B) > 1.$$

Making no further assumptions on $A$ and $B$, prove that $A \cap B \neq \emptyset$.

$r \in [0,1]$

$\mathbb{P}(A \cup B)$

$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$> 1 - \mathbb{P}(A \cap B)$

$r > 1 - \mathbb{P}(A \cap B)$

$\mathbb{P}(A \cap B) > 1 - r \geq 0$

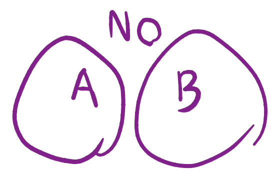
$r \leq 1$

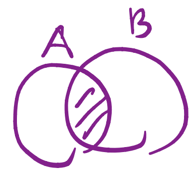
$-r \geq -1$

$1 - r \geq 0$

$\mathbb{P}(A \cap B) > 0.$

$\therefore \quad A \cap B \neq \emptyset$




(b) Prove that $A$ is independent from itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.

$(\Rightarrow)$ Suppose $A$ is indep. from $A$.

$A \cap A = A$

$\mathbb{P}(A) = p$

I.e. $\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$

$\quad\quad \mathbb{P}(A) \quad\quad\quad\quad \mathbb{P}(A)^2$

I.e. $p = p^2 \quad \therefore p - p^2 = 0 \quad \therefore p(1-p) = 0$

$\therefore p = 0$ or $1 - p = 0$

i.e. $p = 1$,

$(\Leftarrow)$ Suppose $\mathbb{P}(A) = 0$ or $1$.

Then $\mathbb{P}(A) = \mathbb{P}(A)^2 \quad (0^2 = 0,\ 1^2 = 1)$

$\quad\quad \mathbb{P}(A \cap A)$ i.e. $A$ is indep. from $A$.

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**2.** Roll two fair dice repeatedly. If the sum is $\geq 10$, then you win.

(a) What is the probability that you start by winning 3 times in a row?

Single round, $\mathbb{P}(\text{success}) = \mathbb{P}(\{(4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\})$

$= \frac{6}{36} = \frac{1}{6}$

$\therefore \mathbb{P}((\text{win}, \text{win}, \text{win})) = \mathbb{P}(\text{win})\mathbb{P}(\text{win})\mathbb{P}(\text{win}) = \left(\frac{1}{6}\right)^3$

$\underline{\underline{\text{Bin}(3, \tfrac{1}{6})}}$

(b) What is the probability that after rolling the pair of dice 5 times you win exactly 3 times?

$X$ = # of wins in 5 trials $\therefore X \sim \text{Bin}(5, \frac{1}{6})$

$\therefore \mathbb{P}(X=3) = \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$

"has prob. distribution ="

(c) What is the probability that the first time you win is before the tenth roll (of the pair), but after the fifth?

$N$ = time of 1st success, $N \sim \text{Geom}(\frac{1}{6})$

$\mathbb{P}(5 < N < 10) = \mathbb{P}(6 \leq N \leq 9)$

$= \mathbb{P}(N=6) + \mathbb{P}(N=7) + \mathbb{P}(N=8) + \mathbb{P}(N=9)$

✓ $\rightarrow$ $= \left(\frac{5}{6}\right)^5\frac{1}{6} + \left(\frac{5}{6}\right)^6\frac{1}{6} + \left(\frac{5}{6}\right)^7\frac{1}{6} + \left(\frac{5}{6}\right)^8\frac{1}{6}$.

$= \left(\frac{5}{6}\right)^5 - \left(\frac{5}{6}\right)^9$

**1.** At a political meeting there are 7 progressives and 7 conservatives. We choose five people uniformly at random to form a committee (president, vice-president and 3 regular members).

(a) Let $A$ be the event that we end up with more conservatives than progressives. What is the probability of $A$?

(b) Let $B$ be the event that Ronald, representing conservatives, becomes the president, and Felix, representing liberals, becomes the vice-president. What is the probability of $B$?

(a) $A_j = \{ j \text{ conservatives are chosen} \}$

$A = A_3 \cup A_4 \cup A_5 \qquad \mathbb{P}(A) = \mathbb{P}(A_3) + \mathbb{P}(A_4) + \mathbb{P}(A_5)$

disjoint

$$\mathbb{P}(A_j) = \frac{\binom{7}{j}\binom{7}{5-j}}{\binom{14}{5}}$$

$$\therefore \mathbb{P}(A) = \frac{\binom{7}{3}\binom{7}{2} + \binom{7}{4}\binom{7}{1} + \binom{7}{5}\binom{7}{0}}{\binom{14}{5}} \qquad \checkmark$$

(b)

$$\frac{\binom{12}{3}}{\binom{14}{5}} = \frac{10}{13 \cdot 7} = \mathbb{P}(\text{Ronald \& Felix are on the committe})$$

$\neq \mathbb{P}(B)$

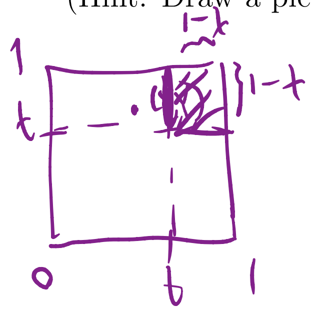
$B = \{ (\text{Ronald}, \text{Felix}, *, *, *) \}$

$$\mathbb{P}(B) = \frac{1 \cdot 1 \cdot 12 \cdot 11 \cdot 10}{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10} = \frac{1}{14 \cdot 13}$$

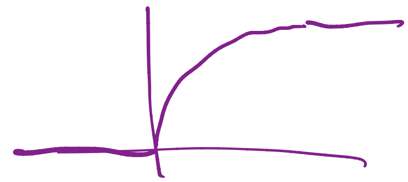
($\frac{1}{20}$th of the above)

**4.** Consider a point $P = (X, Y)$ chosen uniformly at random inside of the unit square $[0,1]^2 = [0,1] \times [0,1] = \{(x, y) : 0 \le x, y \le 1\}$. Let $Z = \min(X, Y)$ be the random variable defined as the minimum of the two coordinates of the point. For example, if $P = (\frac{1}{2}, \frac{1}{3})$, then $Z = \min(\frac{1}{2}, \frac{1}{3}) = \frac{1}{3}$. Determine the cumulative distribution function of $Z$. Determine if $Z$ is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of $Z$. If discrete, determine the probability mass function of $Z$. If neither, explain why.

(Hint: Draw a picture.)




$$F_Z(t) = \begin{cases} 0 & t < 0 \\ 1-(1-t)^2 & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$$



$(0 \le t \le 1)$

$$P(\min(X,Y) \le t) = 1 - P(\min(X,Y) > t)$$
$$= 1 - P(X>t, Y>t)$$
$$= 1 - P((X,Y) \in S)$$

$$S = \{(a,b) : a > t, b > t\}$$

$$P(\{(X,Y) \in S\}) = \frac{\text{Area}(S)}{\text{Area}(\square) = 1} = (1-t)^2$$

$$f_Z(t) = \begin{cases} 0 & t < 0 \\ 2(1-t) & 0 < t \le 1 \\ 0 & t > 1 \end{cases}$$

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$$f_X(t) \cdot 2\varepsilon \approx P(t-\varepsilon \le X \le t+\varepsilon)$$

$$f_X(t) = \lim_{\varepsilon \to 0} \frac{P(t-\varepsilon \le X \le t+\varepsilon)}{2\varepsilon}$$