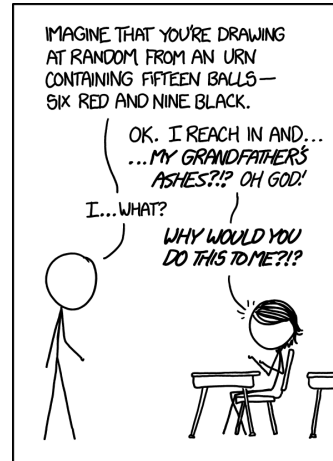


Name: Sol Utians

PID: Aϕ

Seat Number: 1

To affirm your commitment to academic integrity,
write out the phrase "I excel with integrity"



MATH 180A (B00) FINAL EXAM December 9, 2019

INSTRUCTIONS — READ THIS NOW

- Write your name, PID, and current seat number, and write out the academic integrity pledge in the indicated space above. Also, write your name and PID on **every page**. Your score **will not be recorded** if any of this identifying information or the pledge is missing.
- Write within the **boxed areas**, or your work will not be graded. If you need more space, use the back of the page (indicate where). To receive full credit, your answers must be neatly written and logically organized.
- To ask questions during the exam, remain in your seat and raise your hand. Please show your ID to a proctor when you hand in your exam.
- You may not speak to any other student in the exam room while the exam is in progress. You may not share **any information** about the exam with any student who has not yet taken it.
- Turn off and put away all cellphones, calculators, and other electronic devices. You may not access any electronic device during the exam period. You may use your two double-sided pages of hand-written notes, but no other books, notes, or aids.
- This is a 3-hour exam. There are 8 problems on 8 pages, worth a total of 80 points. Read all the problems first before you start working on any of them, so you can manage your time wisely.
- **Have fun!**

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(P. 3)

1. (5 points) I have 8 pairs of socks, each with a different design. I did laundry this morning; I pulled socks out of the dryer one by one, and the first 5 socks I pulled out all had different designs. How likely was that?

8 pairs of sock: 16 socks total.

Sampling without replacement.

$$\rightarrow \Omega = \{(s_1, s_2, s_3, s_4, s_5) : s_i \neq s_j \forall i \neq j, s_j \in \{1, \dots, 16\}\}$$

$$\#\Omega = 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$$

Event in question: (call it S)

choices for s_1 : 16

" " s_2 : 14

" " s_3 : 12

" " s_4 : 10

" " s_5 : 8

$$\#S = 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8$$

$$\therefore P(S) = \frac{\#S}{\#\Omega} = \frac{16 \cdot 14 \cdot 12 \cdot 10 \cdot 8}{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12} = \frac{10 \cdot 8}{15 \cdot 13} = \frac{16}{39}$$

$$\approx 41\%$$

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(P. 5)

2. (10 points) In each scenario below, indicate whether the distribution of X is best described as Binomial, Geometric, Poisson, Exponential, or Normal. A correct answer is worth 2 points; an incorrect answer will get 0 points; a blank answer will get 1 point.

(a) X is the number of phone calls received by a librarian in the next 4 hours.

Binomial Geometric Poisson Exponential Normal

(b) X is the time in minutes before the next pedestrian arrives at a crosswalk.

Binomial Geometric Poisson Exponential Normal

(c) X is the number of 3s you get if you roll a fair die 20 times.

Binomial Geometric Poisson Exponential Normal

(d) X is the number of boxes of cereal you must buy until you get one with your favorite Pokémon figurine inside.

Binomial Geometric Poisson Exponential Normal

(e) X is the score a random college applicant gets on the SAT.

Binomial Geometric Poisson Exponential Normal

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(P. 9)

4. A coin is biased, coming up Heads $\frac{2}{3}$ of the time. You toss the coin 1800 times, and write down the number S of Heads that came up. You'd like to estimate $\mathbb{P}(1150 \leq S \leq 1250)$.

- (a) (5 points) Which is more appropriate to use: the Poisson Approximation or the Normal Approximation? Explain your answer. Use the appropriate approximation to estimate the answer. (You may leave your answer in the form of a sum of Poisson probabilities, or in terms of Φ , as appropriate.)

$$\begin{aligned}
 S &\sim \text{Bin}\left(1800, \frac{2}{3}\right), & np = 1200 \text{ is } \underline{\text{not}} \text{ small; Poisson} \\
 & \uparrow & \text{not approx.} \\
 & n & p \\
 \mathbb{E}(S) &= 1200 & np(1-p) = 1800 \cdot \frac{2}{3} \cdot \frac{1}{3} = 400 \text{ is big;} \\
 \sigma(S) &= \sqrt{400} = 20 & \text{Normal is approx.}
 \end{aligned}$$

$$\begin{aligned}
 \mathbb{P}(1150 \leq S \leq 1250) &= \mathbb{P}(-50 \leq S - 1200 \leq 50) \\
 &= \mathbb{P}\left(-2.5 \leq \frac{S - 1200}{20} \leq 2.5\right) \approx \Phi(2.5) - \Phi(-2.5) \\
 & \quad \uparrow \\
 & \quad \text{standardized} \\
 &= 2\Phi(2.5) - 1 \\
 &= 98.76\%
 \end{aligned}$$

- (b) (5 points) To get a (possibly cruder) estimate without doing much calculation, use Chebyshev's inequality to estimate the same probability. Express your answer as a percent.

$$\begin{aligned}
 \mathbb{P}(1150 \leq S \leq 1250) &= \mathbb{P}\left(|S - 1200| \leq 2.5 \cdot 20\right) \geq 1 - \frac{1}{k^2} \\
 & \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow \\
 & \quad \mu \quad \quad \therefore k \quad \quad \sigma \\
 &= 1 - \frac{1}{(2.5)^2} \\
 &= 1 - \frac{1}{\left(\frac{5}{2}\right)^2} \\
 &= 1 - \frac{4}{25} \\
 &= \frac{21}{25} = 84\%
 \end{aligned}$$

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(P. 11)

5. A random number X is chosen as follows: first, choose a uniformly random number U_1 in the unit interval $[0, 1]$. Then, independently toss a fair coin. If the coin comes up heads, $X = U_1$; if the coin comes up tails, $X = U_1 + 1$. In this exercise, you will show that X is actually uniformly distributed on $[0, 2]$.

(a) (5 points) Let $U_k \sim \text{Unif}[0, k]$ and let $Y \sim \text{Ber}(\frac{1}{2})$. Compute the moment generating functions $M_{U_k}(t)$ and $M_Y(t)$.

$$M_{U_k}(t) = \mathbb{E}(e^{tU_k}) = \int_0^k \frac{1}{k} e^{tx} dx = \frac{1}{kt} (e^{tx}) \Big|_{x=0}^{x=k} = \frac{e^{kt} - 1}{kt}$$

$$\begin{aligned} M_Y(t) &= \mathbb{E}(e^{tY}) = \sum_n e^{tn} P(Y=n) = e^{t \cdot 0} (1 - \frac{1}{2}) + e^{t \cdot 1} (\frac{1}{2}) \\ &= \frac{e^t + 1}{2} \end{aligned}$$

(b) (5 points) Given U_1 and Y as in part (a), independent, compute the moment generating function $M_{U_1+Y}(t)$, and explain why this shows that the distribution of X is $\text{Unif}[0, 2]$.

$$M_{U_1+Y}(t) = M_{U_1}(t) M_Y(t) = \frac{e^t - 1}{t} \cdot \frac{e^t + 1}{2} = \frac{e^{2t} - 1}{2t}$$

↑
independence

↑
recognize this from (a)
as MGF of U_2 .

Since all these MGFs are finite for all t ,

$$M_{U_1+Y} = M_{U_2} \Rightarrow U_1+Y \sim U_2 \text{ as claimed.}$$

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(P. 13)

6. Let (X, Y) be a random vector whose joint density is given by

$$f_{(X,Y)}(x,y) = \begin{cases} xy - 2x & 0 \leq x \leq 1, 2 \leq y \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (6 points) What is the probability that $\frac{X}{Y} \geq \frac{1}{4}$? Set Up But Don't Evaluate

$$P\left(\frac{X}{Y} \geq \frac{1}{4}\right) = P(Y \leq 4X) = P((X,Y) \in T) \text{ where } T = \{(x,y) : y \leq 4x\}$$

$$= \iint (xy - 2x) dx dy$$

$$= \int_2^4 \int_{y/4}^1 x(y-2) dx \quad \text{OR} \quad \int_{1/2}^1 \int_2^{4x} x(y-2) dy$$

$= 7/24$

(b) (5 points) Compute the marginal density f_Y of the random variable Y .

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \begin{cases} 0 & \text{if } y \notin [2,4] \\ \int_0^1 x(y-2) dx & \text{if } y \in [2,4] \end{cases}$$

$$= (y-2) \int_0^1 x dx = \frac{1}{2}(y-2)$$

$$\therefore f_Y(y) = \frac{1}{2}(y-2) \mathbb{1}_{\{2 \leq y \leq 4\}}$$

(c) (4 points) Are X and Y independent? Explain.

Yes. $f_{(X,Y)}(x,y) = x(y-2) \mathbb{1}_{\{0 \leq x \leq 1, 2 \leq y \leq 4\}}$

$$= x \mathbb{1}_{\{0 \leq x \leq 1\}} \cdot (y-2) \mathbb{1}_{\{2 \leq y \leq 4\}}$$

is a product $u(x) \cdot v(y)$. $\therefore X, Y$ are independent.

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7. Suppose X and Y are integer-valued random variables whose joint probability mass function is given by $\mathbb{P}(X = j, Y = k) = \frac{1}{6}$ if $1 \leq j \leq k \leq 3$ and $\mathbb{P}(X = j, Y = k) = 0$ otherwise.

(a) (6 points) Compute the probability mass function of X and the probability mass function of Y . (You may find it convenient to use a chart.) Are X and Y independent? Justify your answer.

	1	2	3	$\rightarrow X$
1	$\frac{1}{6}$	0	0	$\frac{1}{6}$
2	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
$\downarrow Y$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	

marginal Y

k	$P_X(k)$	$P_Y(k)$
1	$\frac{1}{2}$	$\frac{1}{6}$
2	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{1}{6}$	$\frac{1}{2}$

$\mathbb{P}(X=2)\mathbb{P}(Y=1) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$

$\mathbb{P}(X=2, Y=1) = 0 \rightarrow$ NOT =

$\therefore X, Y$ not independent.

(b) (4 points) Compute $\mathbb{P}(X = Y - 1)$.

$$\begin{aligned}
 \mathbb{P}(X = Y - 1) &= \sum_{k=1}^3 \mathbb{P}(X = Y - 1, Y = k) = \sum_{k=1}^3 \mathbb{P}(X = k - 1, Y = k) \\
 &= \mathbb{P}(X = 0, Y = 1) \\
 &\quad + \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 2, Y = 3) \\
 &= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.
 \end{aligned}$$

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(P. 17)

8. Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables, each with a normal $\mathcal{N}(1, 1)$ distribution. Let $T_n = X_1^2 + X_2^2 + \dots + X_n^2$.

(a) (5 points) Show that there is a real number c such that $\lim_{n \rightarrow \infty} \frac{T_n}{n} = c$ with probability 1, and find the value of c .

$$X_j \sim \mathcal{N}(1, 1); \therefore X_j = 1 + Z_j \text{ where } Z_j \sim \mathcal{N}(0, 1).$$

$$\therefore \mathbb{E}(X_j^2) = \mathbb{E}((1 + Z_j)^2) = \mathbb{E}(1 + 2Z_j + Z_j^2) = 1 + 2\underbrace{\mathbb{E}(Z_j)}_0 + \mathbb{E}(Z_j^2)$$

$$= 2.$$

$$\therefore \text{By SLLN, } \frac{T_n}{n} \xrightarrow{\text{w.p. } 1} \mathbb{E}\left(\frac{T_n}{n}\right) = \frac{1}{n}(2 + 2 + \dots + 2) = \frac{2}{n} \cdot n = 2 \equiv c$$

(b) (5 points) Let c be the constant from part (a). What is $\lim_{n \rightarrow \infty} \mathbb{P}(T_n \leq cn)$?

[Hint: Standardize T_n .]

$$\mathbb{P}(T_n \leq 2n) = \mathbb{P}(T_n - 2n \leq 0) = \mathbb{P}\left(\frac{T_n - 2n}{\sqrt{n\sigma}} \leq 0\right)$$

$$\mathbb{E}(T_n) = 2n.$$

$$\text{Var}(T_n) = \text{Var}(X_1^2) + \dots + \text{Var}(X_n^2)$$

$$= n \cdot \sigma^2 \leftarrow \text{could calculate, but don't need to.}$$

$$\xrightarrow[n \rightarrow \infty]{\text{CLT}} \Phi(0) = \frac{1}{2}$$

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