THE WEIERSTRASS PATHOLOGICAL FUNCTION

Until Weierstrass published his shocking paper in 1872, most of the mathematical world (including luminaries like Gauss) believed that a continuous function could only fail to be differentiable at some collection of isolated points. In fact, it turns out that "most" continuous functions are non-differentiable at *all* points. (To understand what this statement could mean, you should take courses in topology and measure theory.) However, Weierstrass was not, in fact, the first to construct such a pathological function. He was preceded by Bolzano (in 1830), Cellérier (also 1830), and Riemann (1862). None of the others published their work (indeed, their examples were not discovered in their notes until after their deaths).

All known examples of non-differentiable smooth functions are constructed in a similar fashion to the following example – they are limits of functions that oscillate more and more on small scales, but with higher-frequency oscillations being damped quickly. The example we give here is a faithful reproduction of Weierstrass's original 1872 proof. It is somewhat more complicated than the example given as Theorem 7.18 in Rudin, but is superior in at least one important way, as explained in Remark 0.2.

THEOREM 0.1. *Let $0 < a < 1$, and choose a positive odd integer b large enough that $\frac{\pi}{ab-1} < \frac{2}{3}$ (i.e. $ab > 1 + \frac{3\pi}{2}$). For example, take $a = \frac{1}{2}$ and $b = 11$. Define the function $W : \mathbb{R} \to \mathbb{R}$ by*

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x).$$

Then W is uniformly continuous on $\mathbb{R}$, but is differentiable at no point.

REMARK 0.2. Notice that the partial sums $W_N(x) = \sum_{n=0}^{N} a^n \cos(b^n \pi x)$ are all C^∞ functions. As the following proof shows, these partial sums converge uniformly to W, and so we have an example here of a sequence of C^∞ functions that converge uniformly to a nowhere-differentiable function. This is the most dramatic demonstration that differentiability is *not* preserved under uniform convergence! The example of Theorem 7.18 in Rudin, while similar in spirit, constructs a function as a uniformly convergent series of functions that have sharp cusps on ever-denser sets, not achieving the same demonstration. Indeed, from Rudin's proof one might be left with the impression that the construction depends fundamentally on the non-smooth points in the approximating functions. As the proof below shows, this has nothing to do with the reason the limit function is non-smooth. The real reason is oscillation on small scales, and this can be achieved with smooth oscillations of high frequency.

PROOF. First note that, since $|\cos(b^n \pi x)| \leq 1$ for all x, we have the terms in the summation are bounded by a^n, and $\sum_n a^n$ is absolutely convergent. Therefore, by the Weierstrass M-test (Theorem 7.10 in Rudin), the sum $W(x)$ converges uniformly. Since the partial sums are all C^∞ (as explained in Remark 0.2), they are continuous, and so their uniform limit W is continuous on $\mathbb{R}$ by Theorem 7.12 in Rudin.

We will now show that, at any arbitrary point $x_0 \in \mathbb{R}$, W is not differentiable at x_0. The strategy is as follows: we construct two sequences (x_m^+) and (x_m^-), such that $x_m^+ \to x_0$ from the right and $x_m^- \to x_0$ from the left, and such that the difference quotients

$$D_m^{\pm} f = \frac{f(x_m^{\pm}) - f(x_0)}{x_m^{\pm} - x_0}$$

do not have the same limit. In fact, we will show that $|D_m^{\pm} f|$ diverges to ∞ as $m \to \infty$, and that the two have opposite signs. So on small scales, f oscillates infinitely-often with infinite slope! Figure 1 pictures this function.

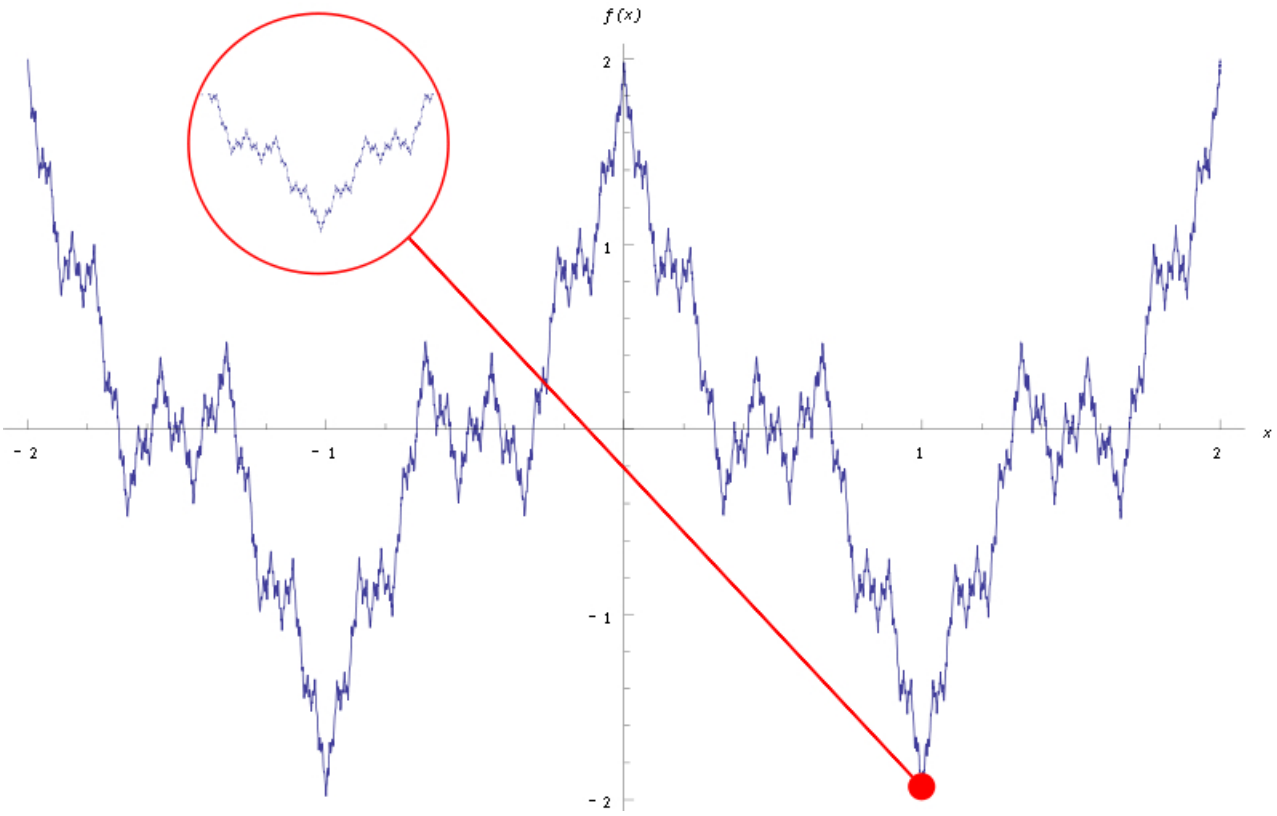

FIGURE 1. The graph of the Weiestrass function W.

Fix $x_0 \in \mathbb{R}$. For each $m \in \mathbb{N}$, we can choose an integer $\alpha_m \in \mathbb{Z}$ such that

$$b^m x_0 = \alpha_m + \epsilon_m$$

where $\epsilon_m \in [-\frac{1}{2}, \frac{1}{2})$. ($\alpha_m$ is either $\lfloor b^m x_0 \rfloor$ or $\lfloor b^m x_0 \rfloor - 1$, depending whether the fractional part of $b^m x_0$ is $\leq \frac{1}{2}$ or $> \frac{1}{2}$.) Now, define

$$x_m^{\pm} \equiv \frac{\alpha_m \pm 1}{b^m} = \frac{b^m x_0 - \epsilon_m \pm 1}{b^m} = x_0 + \frac{\pm 1 - \epsilon_m}{b^m}.$$

Since $|\pm 1 - \epsilon_m| \leq 2$, and $b \geq 3$, we see that $x_m^{\pm} - x_0$ converges to 0. Notice also from the signs that $x_m^- < x_0 < x_m^+$. We now examine the difference quotients,

$$D_m^{\pm} f = \frac{f(x_m^{\pm}) - f(x_0)}{x_m^{\pm} - x_0} = \sum_{n=0}^{\infty} a^n \frac{\cos(b^n \pi x_m^{\pm}) - \cos(b^n \pi x_0)}{x_m^{\pm} - x_0}.$$

(We can subtract the sums because they converge, by the limit theorems.) We now break up the sum into two parts, with n ranging between 0 and $m-1$, and from m on up.

$$D_m^{\pm} f = \sum_{n=0}^{m-1} a^n \frac{\cos(b^n \pi x_m^{\pm}) - \cos(b^n \pi x_0)}{x_m^{\pm} - x_0} + \sum_{n=m}^{\infty} a^n \frac{\cos(b^n \pi x_m^{\pm}) - \cos(b^n \pi x_0)}{x_m^{\pm} - x_0}.$$

Refer to these two sums as $S_1^{\pm} + S_2^{\pm}$. First, we bound the first one. Rewriting the terms as

$$a^n \frac{\cos(b^n \pi x_m^{\pm}) - \cos(b^n \pi x_0)}{x_m^{\pm} - x_0} = (ab)^n \pi \frac{\cos(b^n \pi x_m^{\pm}) - \cos(b^n \pi x_0)}{b^n \pi x_m^{\pm} - b^n \pi x_0},$$

we see this is of the form $(ab)^n \pi \frac{\cos(A)-\cos(B)}{A-B}$. By the mean value theorem, there is a point C between A and B where $\frac{\cos(A)-\cos(B)}{A-B} = -\sin C$, and the absolute value is therefore ≤ 1. Hence, we have

$$|S_1^{\pm}| \leq \sum_{n=0}^{m-1} (ab)^n \pi = \pi \frac{(ab)^m - 1}{ab - 1} < \pi \frac{(ab)^m}{ab - 1}. \tag{0.1}$$

Turning now to the second term, we reindex $k = n - m$,

$$S_2^{\pm} = \sum_{k=0}^{\infty} a^{k+m} \frac{\cos(b^{k+m} \pi x_m^{\pm}) - \cos(b^{k+m} \pi x_0)}{x_m^{\pm} - x_0}. \tag{0.2}$$

The argument of $\cos$ in the first term is

$$\cos(b^{k+m} \pi x_m^{\pm}) = \cos\left(b^k \pi \cdot b^m \frac{\alpha_m \pm 1}{b^m} \right) = \cos\left(b^k (\alpha_m \pm 1) \pi \right).$$

Since b^k is an odd integer and $\alpha \pm 1$ is an integer, this number is ± 1, with the sign determined by the parity of $\alpha_m \pm 1$; that is,

$$\cos(b^{k+m} \pi x_m^{\pm}) = (-1)^{\alpha_m \pm 1} = -(-1)^{\alpha_m}.$$

The argument of $\cos$ in the second term in Equation 0.2 is

$$b^{k+m} \pi x_0 = b^{k+m} \pi \frac{\alpha_m + \epsilon_m}{b^m} = b^k \pi (\alpha_m + \epsilon_m).$$

We now use that summation formula for cosine, $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$, so that

$$\cos(b^{k+m} \pi x_0) = \cos(b^k \alpha_m \pi) \cos(b^k \epsilon_m \pi) - \sin(b^k \alpha_m \pi) \sin(b^k \epsilon_m \pi).$$

Since $b^k \alpha_m \in \mathbb{Z}$, the second term is 0. Also, we have $\cos(b^k \alpha_m \pi) = (-1)^{\alpha_m}$. So

$$\cos(b^{k+m} \pi x_0) = (-1)^{\alpha}_{m} \cos(b^k \epsilon_m \pi).$$

Combining this with Equation 0.2, we have

$$S_2^{\pm} = \sum_{k=0}^{\infty} a^{k+m} \frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m} \cos(b^k \epsilon_m \pi)}{x_m^{\pm} - x_0}. \tag{0.3}$$

Using the fact that $x_m^{\pm} - x_0 = (\pm 1 - \epsilon_m)/b^m$, we can now simplify Equation 0.3 to read

$$S_2^{\pm} = (ab)^m (-1)^{\alpha_m} \sum_{k=0}^{\infty} \frac{1 + \cos(b^k \epsilon_m \pi)}{\epsilon_m \mp 1}. \tag{0.4}$$

We now consider the two sums, S_2^+ and S_2^- separately. From Equation 0.4 we have

$$\frac{(-1)^{\alpha_m}}{(ab)^m} S_2^+ = \sum_{k=0}^{\infty} \frac{1 + \cos(b^k \epsilon_m \pi)}{\epsilon_m - 1}.$$

Since $\epsilon_m \leq \frac{1}{2}$, and since $\cos(b^k \epsilon_m \pi) \geq -1$, all the terms in this sum are ≤ 0. Hence, the negative of this summation has non-negative terms, and so is bounded below by the $k = 0$ term:

$$-\frac{(-1)^{\alpha_m}}{(ab)^m} S_2^+ \geq \frac{1 + \cos(\epsilon_m \pi)}{1 - \epsilon_m}.$$

Since $\epsilon_m > -\frac{1}{2}$, $\frac{1}{1-\epsilon_m} \geq \frac{1}{1+\frac{1}{2}}$. Since $\epsilon_m \pi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, its cosine is ≥ 0, and the numerator is ≥ 1. Hence, we have

$$-\frac{(-1)^{\alpha_m}}{(ab)^m} S_2^+ \geq \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}. \tag{0.5}$$

Now we consider S_2^-. This time Equation 0.4 yields that

$$\frac{(-1)^{\alpha_m}}{(ab)^m} S_2^- = \sum_{k=0}^{\infty} \frac{1 + \cos(b^k \epsilon_m \pi)}{e_m + 1}$$

is a sum of non-negative terms, and precisely the same analysis as above demonstrates that

$$\frac{(-1)^{\alpha_m}}{(ab)^m} S_2^- \geq \frac{2}{3}. \tag{0.6}$$

We're now ready to finish the proof. For we have, combining Equations 0.1 and 0.5,

$$(ab)^{-m} D_m^+ f = (ab)^{-m} S_1^+ + (ab)^{-m} S_2^+,$$

where

$$|(ab)^{-m} S_1^+| \leq \frac{\pi}{ab - 1}, \qquad -(-1)^{\alpha_m} (ab)^{-m} S_2^+ \geq \frac{2}{3}.$$

To ease notation, let $T_j^{\pm} = (ab)^{-m} S_j^{\pm}$ for $j = 1, 2$; then

$$|T_1^+| \leq \frac{\pi}{ab - 1}, \qquad -(-1)^{\alpha_m} T_2^+ \geq \frac{2}{3}.$$

Thus, $T_1^+ \in [\frac{\pi}{ab-1}, \frac{\pi}{ab-1}]$, while T_2^+ is a number (positive or negative) outside the interval $(-\frac{2}{3}, \frac{2}{3})$. By assumption, $\frac{\pi}{ab-1} < \frac{2}{3}$, and so $(ab)^{-m} D_m^+ f = T_1^+ + T_2^+$ is a number with absolute value bigger than $\frac{2}{3} - \frac{\pi}{ab-1}$. In other words, as $m \to \infty$, $(ab)^{-m} D_m^+ f$ does not tend to 0. Since $(ab)^m \to \infty$ as $m \to \infty$, this proves that the right-derivative of f does not exist at x_0 – in absolute value, it blows up to ∞. This is enough to prove theorem.

Note also that a similar analysis shows that

$$|T_1^-| \leq \frac{\pi}{ab - 1}, \qquad (-1)^{\alpha_m} T_2^- \geq \frac{2}{3}.$$

Hence $D_m^- f$ also blows up in absolute values as we approach x_0 from the left. But the interesting point here is that T_2^+ and T_2^- have *opposite signs*, which means that not only is the function non-differentiable at each point, but approaching from left and right we have slopes increasing without bound and with opposite sign. So the function oscillates badly on all scales at all points.

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