MATH 140B: FOUNDATIONS OF REAL ANALYSIS II
SUMMARY OF KEY FACTS FOR EXAM 2

TODD KEMP

Throughout, reference numbers (like Definition 2.18 and Proposition 2.21) refer to the course lecture notes, also available on the website.

Differentiation of Functions of a Real Variable

(1) L'Hôpital's Rule (Theorem 8.25).
(2) Extended Mean Value Theorem (Lemma 8.26).
(3) Derivatives of vector-valued function (Definition 8.28), in terms of derivatives of components (Proposition 8.29).
(4) Mean Value Theorem and L'Hôpital's Rule fail for vector-valued functions (Examples 8.31, 8.32).
(5) Mean Value Inequality for vector-valued functions (Theorem 8.33).
(6) The Cauchy-Schwarz Inequality (Lemma 8.34).

Integration

(1) Partitions, upper and lower sums, upper and lower integrals (Definitions 9.1-9.3).
(2) Dirichlet's function ($\mathbb{1}_Q$) is not Riemann integrable (Example 9.6).
(3) Riemann–Stieltjes (Darboux) upper and lower sums, integrals $L(f, \alpha)$ and $U(f, \alpha)$ (Definition 9.7).
(4) $L(f, \alpha) \leq U(f, \alpha)$ (Proposition 9.8).
(5) Quantitative (ϵ–Π) statement of Riemann–Stieltjes integrability (Lemma 9.10).
(6) Continuous functions are always in $\mathscr{R}(\alpha)$ (Theorem 9.11).
(7) Finitely-many discontinuities at continuous points of α also yields $\mathscr{R}(\alpha)$ (Theorem 9.12).
(8) Basic properties of the integral (Lemmas 9.16–9.23).
(9) "Delta function" integrators (Example 9.24, Lemma 9.25).
(10) If α is differentiable and $\alpha' \in \mathscr{R}$ then $\int f\, d\alpha = \int f\alpha'$ for $f \in \mathscr{R}(\alpha)$ (Theorem 9.27).
(11) The Change of Variables formula (Theorem 9.28).
(12) The Fundamental Theorem of Calculus (Theorem 9.30).
(13) Integration by Parts (Theorem 9.31).
(14) Rectifiable curves (Definition 9.32 and following paragraphs).
(15) Vector-valued integration and inequalities (Definition 9.34, Lemma 9.35).
(16) C^1 curves are rectifiable (Theorem 9.37).

Sequences and Series of Functions

(1) Pointwise convergence (Definition 10.1).
(2) Pointwise convergence doesn't respect continuity, smoothness, integration (Examples 10.2-10.4).
(3) Uniform convergence (Definition 10.6).

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(4) Uniform convergence is equivalent to uniform Cauchy-ness (Proposition 10.8).
(5) Uniform convergence preserves limits and continuity (Theorem 10.9).
(6) The metric space $C_b(X)$ of bounded continuous $\mathbb{R}$-valued functions on a metric space X, equipped with the uniform metric $d_u(f,g) = \sup_{x\in X}|f(x) - g(f)|$, is a complete metric space (Corollary 10.10).
(7) Dini's Monotone Convergence Theorem (Theorem 10.13).
(8) Uniform convergence preserves Riemann–Stieltjes integrals (Theorem 10.16).
(9) Uniform convergence preserves *anti*derivatives (Theorem 10.19).
(10) Uniformly convergent series of functions (Definition 10.20).
(11) Weierstrass M-test (Lemma 10.23).