Homework 1 Math280C Spring 2018

Due Friday in class, April 14. Relevant sections in Durrett's textbook 8.1 and 8.2. Justify all your answers.

Through out this homework $(B_t)_{t\geq 0}$ denotes standard Brownian motion.

1. Show that almost surely there does not exist any time interval (a, b) with $0 \leq a < b$ such that $t \to B_t$ is either strictly increasing or strictly decreasing on (a, b). That is, show that almost surely Brownian motion is not monotone on any time interval.

2. Show that if $\gamma > 1/2$, then the Brownian path B_t is not Hölder continuous with exponent γ at any point in $[0, 1]$.

3. Fix $t > 0$. For $n \in \mathbb{N}$ and $1 \leq m \leq 2n$, let $\Delta_{m,n} = B_{tm2^{-n}} - B_{t(m-1)2^{-n}}$.

(a) Show that

$$\mathbb{E}\left[\sum_{m=1}^{2^n} \Delta_{m,n}^2\right] = t.$$

(b) Use Borel-Cantelli to show that as $n \to \infty$,

$$\sum_{m=1}^{2^n} \Delta_{m,n}^2 \to t \quad a.s.$$

4. Fix $\lambda > 0$. Let

$$X_t = e^{-\lambda t} B_{e^{2\lambda t}}.$$

A stochastic process with continuous paths and the same finite-dimensional distributions as $(X_t)_{t\in\mathbb{R}}$ is called an Ornstein-Uhlenbeck process.

(a) Show that Xt has a standard normal distribution for all $t \in \mathbb{R}$.

(b) Calculate covariance function $Cov(X_s, X_t)$ for all $s, t \in \mathbb{R}$.

(c) Show that the Ornstein-Uhlenback process is stationary: that is for any $s \in \mathbb{R}$, the process $(Y_t)_{t\in\mathbb{R}}$ defined as $Y_t = X_{t+s}$ has the FDDs as $(X_t)_{t\in\mathbb{R}}$.

5. Let $(X_t)_{t\in\mathbb{R}}$ be an Ornstein-Uhlenbeck process as above. Show that $(X_t)_{t\geq 0}$ is a Markov process and write down its Markov transition kernels.