CONSTRUCTING A GRÖBNER BASIS OF GRIFFIN'S IDEAL

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ABSTRACT. In his Ph.D. thesis, Sean Griffin introduced a family of ideals and found monomial bases for their quotient rings. These rings simultaneously generalize the Delta Conjecture coinvariant rings of Haglund-Rhoades-Shimozono and the cohomology rings of Springer fibers studied by Tanisaki and Garsia-Procesi. We recursively construct a Gröbner basis of Griffin's ideals with respect to the graded reverse lexicographical order. Consequently, Griffin's monomial basis is the standard monomial basis. Coefficients of polynomials in our Gröbner basis are integers and leading coefficients are one.

1. INTRODUCTION

For $n \in \mathbb{Z}_{>0}$, a partition λ, and $s \in \mathbb{Z}_{>0} \sqcup \{\infty\}$ with $1 \leqslant |\lambda| \leqslant n$ and $s \geqslant \ell(\lambda)$, Griffin introduced ideals $I_{n,\lambda,s}$ in the polynomial ring $\mathbb{Q}[x_1, \cdots, x_n]$. Let $R_{n,\lambda,s} := \mathbb{Q}[x_1, \cdots, x_n]/I_{n,\lambda,s}$ be their corresponding quotient ring. The ideal $I_{n,\lambda,s}$ and the ring $R_{n,\lambda,s}$ generalize the coinvariant ideal $I_n := I_{n,(1)^n,\infty}$ and the coinvariant ring $R_n := R_{n,(1)^n,\infty}$, where $(1)^n$ is the partition with n copies of 1. The coinvariant ring R_n presents the cohomology of complete flag variety in type A_{n-1}. The ideal I_n and the ring R_n have two other generalizations, both of which can be recovered as special cases of $I_{n,\lambda,s}$ and $R_{n,\lambda,s}$.

- For a partition λ, $I_{|\lambda|,\lambda,\infty}$ is the Tanisaki ideal I_λ and $R_{|\lambda|,\lambda,\infty}$ is the Tanisaki quotient R_λ studied by Tanisaki [Tan82] and Garsia-Procesi [GP92]. The ring R_λ presents the cohomology of the Springer fiber of λ.
- For some $k \leqslant n$ and $k \leqslant s < \infty$, the $I_{n,(1)^k,s}$ and $R_{n,(1)^k,s}$ correspond to the $I_{n,k,s}$ and $R_{n,k,s}$ introduced by Haglund-Rhoades-Shimozono [HRS18]. The ring $R_{n,k,s}$ gives a representation-theoretic model for the Haglund-Remmel-Wilson Delta Conjecture [HRW18]. Pawlowski and Rhoades proved that $R_{n,k,s}$ presents the cohomology of n-tuples $(\ell_1, ..., \ell_n)$ of lines in $\mathbb{C}^n$ such that the projection of $\ell_1 + ... + \ell_n$ onto $\mathbb{C}^k$ is surjective.

The theory of Gröbner basis plays a crucial role in the study of ideals in polynomial rings. Most computation problems, such as determining whether a given polynomial is in the ideal or finding generators of intersections of ideals, can be answered with Gröbner bases. Moreover, a Gröbner basis can be used to find a monomial basis of the quotient ring, known as the standard monomial basis. Haglund, Rhoades and Shimozono found the reduced Gröbner basis of $I_{n,k,s}$ to obtain a monomial basis of $R_{n,k,s}$.

In this paper, we find a Gröbner basis of $I_{n,\lambda,s}$ with respect to the graded reverse lexicographical order. To the best of the author's knowledge, finding a Gröbner basis for even Tanisaki ideals I_λ remained open. The form of our Gröbner basis implies Griffin's monomial basis of $R_{n,\lambda,s}$ in [Gri20] is the standard monomial basis. One notable feature of our Gröbner basis is that its polynomials have integer coefficients with leading coefficient one. This is rarely the case, even for ideals with nice generating sets and important ties to algebra, geometry and combinatorics.

The Gröbner bases we construct are not minimal thus not reduced. However, since each polynomial has integer coefficients with leading coefficient 1, we can deduce the polynomials in the reduced Gröbner bases also have integer coefficients. We believe our work is a good starting point for addressing the following open problem:

Problem 1.1. Find the reduced Gröbner bases of all $I_{n,\lambda,s}$.

The rest of the paper is structured as follows. In §2, we provide necessary background on Griffin's ideals and Gröbner bases. We then deduce that a Gröbner basis of $I_{n,\lambda,s}$ when $s<\infty$ can be obtained by adding $x_1^s,\cdots,x_n^s$ to a Gröbner basis of $I_{n,\lambda,\infty}$. In §3, we find a finite set of monomials $B_{n,\lambda}$ that generates the initial ideal of $I_{n,\lambda,\infty}$. Our main tool is a combinatorial construction called container diagram developed by Rhoades, Yu and Zhao [RYZ20]. Finally, in §4, we recursively construct a polynomial in $I_{n,\lambda,\infty}$ with leading monomial x^α for each $x^\alpha\in B_{n,\lambda}$. Consequently, the polynomials we construct form a Gröbner basis of $I_{n,\lambda,\infty}$.

2. Background

2.1. **Defining $I_{n,\lambda,s}$ and $R_{n,\lambda,s}$.** Given $S\subseteq[n]:=\{1,\cdots,n\}$, let $e_d(S)$ be the *elementary symmetric function* of degree d with variable set $\{x_i : i\in S\}$:

$$e_d(S):=\sum_{\{i_1<i_2<\cdots<i_d\}\subseteq S} x_{i_1}x_{i_2}\cdots x_{i_d}.$$

By convention, $e_0(S)=1$ for any S and $e_d(S)=0$ if $d>|S|$ or $d<0$. We recall one simple but useful identity:

$$e_d(S\sqcup\{1\})=e_d(S)+x_1e_{d-1}(S), \tag{1}$$

where d is any integer and S is a set not containing 1.

A *partition* λ is a weakly decreasing sequence of non-negative integers with only finitely many positive entries. When writing down a partition, we often ignore the trailing 0s. We denote its i^{th} entry of λ by λ_i. Let $\ell(\lambda)$ and $|\lambda|$ denote the number of positive entries and the sum of entries in λ respectively. We say λ is a partition of n if $n=|\lambda|$. The *conjugate* of λ, denoted as λ', is a partition with $\lambda'_i=|j:\lambda_j\geqslant i|$. Following Griffin's convention, for $n\geqslant m$, we define $p^n_m(\lambda):=\lambda'_{n-m+1}+\lambda'_{n-m+2}+\cdots$.

Example 2.1. If $\lambda=(3,2)$, we have $\lambda'=(2,2,1)$. Say $n=7$, then $p^7_7(\lambda)=2+2+1=5$, $p^7_6(\lambda)=2+1=3$, $p^7_5(\lambda)=1$, and $p^7_m(\lambda)=0$ if $m<5$.

Griffin's ideal $I_{n,\lambda,s}$ and Griffin's ring $R_{n,\lambda,s}$ can be defined as follows.

Definition 2.2. [Gri20, Definition 3.0.1] Take a number n. Let λ be a partition with $1\leqslant|\lambda|\leqslant n$. If $s=\infty$, $I_{n,\lambda,s}$ is the ideal of $\mathbb{Q}[x_1,\cdots,x_n]$ generated by:

$$e_d(S) \text{ for all } S\subseteq[n], d>|S|-p^n_{|S|}(\lambda).$$

Now if s is a number such that $s\geqslant\ell(\lambda)$. Then $I_{n,\lambda,s}$ is the ideal generated by the generators of $I_{n,\lambda,\infty}$ above together with $x_1^s,\cdots,x_n^s$.

Finally, let $R_{n,\lambda,s}:=\mathbb{Q}[x_1,\cdots,x_n]/I_{n,\lambda,s}$ for all $\ell(\lambda)\leqslant s\leqslant\infty$.

Example 2.3. Following Example 2.1, we let $n=7$, $\lambda=(3,2)$. Then $I_{n,\lambda,\infty}$ is generated by $e_d(S)$ where d and $S\subseteq[7]$ can take the following values.

- If $|S|=7$, $d>|S|-5=2$.
- If $|S|=6$, $d>|S|-3=3$.
- If $|S|=5$, $d>|S|-1=4$.
- If $|S|<5$, $d>|S|-0=|S|$, forcing $e_d(S)$ to vanish.

The ideal $I_{n,\lambda,5}$ is generated by the generators above together with $x_1^5,\cdots,x_7^5$.

Remark 2.4. We may slightly extend the definition of $I_{n,\lambda,s}$ and $R_{n,\lambda,s}$. When $|\lambda|>n$, we still define $I_{n,\lambda,s}$ and $R_{n,\lambda,s}$ in the same way as above. In this case, $p^n_n(\lambda)=|\lambda|>n$, so $e_0([n])$ is a generator of $I_{n,\lambda,s}$. Notice that $e_0([n])=1$, so $I_{n,\lambda,s}$ is just $\mathbb{Q}[x_1,\cdots,x_n]$ and $R_{n,\lambda,s}$ is the trivial ring. Our extension does not introduce anything interesting, but makes some of our recursive arguments easier to state.

2.2. **Gröbner basis.** We recall some fundamental background regarding the Gröbner basis. A *monomial order* is a total order $<$ on the monomials of $x_1, x_2, \cdots, x_n$ such that

- For any monomial m, $1 \leqslant m$, and
- For any three monomials m_1, m_2 and m_3, $m_1 < m_2$ implies $m_1 m_3 < m_2 m_3$.

Let $<$ be a monomial order. For f in $\mathbb{Q}[x_1, \cdots, x_n]$, its *leading monomial* with respect to $<$, denote as $\mathsf{in}_<(f)$, is the largest monomial with non-zero coefficient in f. For an ideal I of $\mathbb{Q}[x_1, \cdots, x_n]$, its *initial ideal*, denoted as $\mathsf{in}_<(I)$, is the ideal generated by $\mathsf{in}_<(f)$ for $f \in I - \{0\}$. Then $\{g_1, \ldots, g_N\} \subseteq I$ is a *Gröbner basis* of I if $\langle \mathsf{in}_<(g_1), \ldots, \mathsf{in}_<(g_N) \rangle = \mathsf{in}_<(I)$. This implies that $g_1, \ldots, g_N$ generate I. Furthermore, we say $\{g_1, \ldots, g_N\}$ is a *reduced Gröbner basis* if

- The coefficient of $\mathsf{in}_<(g_i)$ in g_i is 1, and
- For $i \neq j$, $\mathsf{in}_<(g_i)$ does not divide any monomial with a non-zero coefficient in g_j.

Every ideal I has exactly one reduced Gröbner basis with respect to a given monomial order.

One application of Gröbner bases is to find a monomial basis of the quotient ring. Consider an ideal I and some monomial order $<$. A monomial m is called a *standard monomial* if m is not in $\mathsf{in}_<(I)$. It is known that $\{m + I : m \text{ is a standard monomial}\}$ forms a basis of $\mathbb{Q}[x_1, \cdots, x_n]/I$. This basis is called the *standard monomial basis*. Notice that this basis is closed under taking factor: If $m + I$ is in this basis and let m' be a factor of m, then $m' + I$ is also in this basis. Determine whether a monomial is standard can be a painful process since one needs to determine whether a monomial is in an ideal. The Gröbner basis can make this process painless. Let G be a Gröbner basis of I with respect to $<$. A monomial m is a standard monomial if and only if $\mathsf{in}_<(g)$ does not divide m for any $g \in G$.

One classical monomial order is the *lexicographical order* $<_{lex}$. We may define this order as $x_1^{a_1} x_2^{a_2} \cdots <_{lex} x_1^{b_1} x_2^{b_2} \cdots$ if there exists i such that $a_j = b_j$ for $j > i$ and $a_j < b_j$.[1] A Gröbner basis can be hard to find and its polynomials rarely have integer coefficients. Nevertheless, the reduced Gröbner bases of I_n, and more generally $I_{n,k,s}$, with respect to the lexicographical order are well-behaved classical polynomials. It is a well-known result that

$$\{h_d([n-d+1]) : d \in [n]\}, \text{ where } h_d([n-d+1]) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant n-d+1} x_{i_1} \cdots x_{i_d}$$

is the reduced Gröbner basis of I_n (see [Stu08, Theorem 1.2.7] and [Ber09, Sec 7.2]). Haglund, Rhoades and Shimozono extend this result. In [HRS18, Theorem 4.14], they show the reduced Gröbner basis of $I_{n,k,s}$ consists of certain *key polynomials* together with $x_1^s, \cdots, x_n^s$ when $k < s$. A notable feature of this reduced Gröbner basis is that its polynomials all have positive integer coefficients.

This paper considers a different monomial order.

Definition 2.5. [CLOS94, §2 Definition 6] Define the *graded reverse lexicographical order* $<_{grevlex}$ as $x_1^{a_1} x_2^{a_2} \cdots <_{grevlex} x_1^{b_1} x_2^{b_2} \cdots$ if either of the following holds

- The degree of $x_1^{a_1} x_2^{a_2} \cdots$ is less than the degree of $x_1^{b_1} x_2^{b_2} \cdots$, or
- They have the same degree and there exists i such that $a_j = b_j$ for all $1 \leqslant j < i$ and $a_j > b_j$.

After switching to this order, the reduced Gröbner bases mentioned above remain to be reduced Gröbner bases.

- For each $h_d([n-d+1])$, its leading monomial is x_{n-d+1}^d with respect to either $<_{lex}$ or $<_{grevlex}$. Consequently, $\{h_d([n-d+1]) : d \in [n]\}$ is still the reduced Gröbner basis of I_n with respect to $<_{grevlex}$.

[1] To make our statements concise, we reverse the usual definition of monomial orders in this paper. Usually, the "$j > i$" in the definition of lexicographical order is replaced by $j < i$.

- Furthermore, the leading monomial of every key polynomial is the same with respect to $<_{lex}$ or $<_{grevlex}$. Thus, the reduced Gröbner basis found by Haglund, Rhoades and Shimozono is also the reduced Gröbner basis of $I_{n,k,s}$ with respect to $<_{grevlex}$.

We conjecture this pattern holds in general:

Conjecture 2.6. *The reduced Gröbner basis of $I_{n,\lambda,s}$ is the same for $<_{grevlex}$ or $<_{lex}$.*

This conjecture has been computer checked for all $n \leqslant 7$ and $s \leqslant 7$.

Hereafter, we use $<$ to denote the graded reverse lexicographical order, in place of $<_{grevlex}$.

2.3. **Griffin's monomial basis of $R_{n,\lambda,s}$.** Before constructing the Gröbner basis, we need to know what monomials are in the initial ideal $\mathsf{in}_<(I_{n,\lambda,s})$. We may predict the structure of $\mathsf{in}_<(I_{n,\lambda,s})$ from Griffin's study of $R_{n,\lambda,s}$. First, recall a few definitions in Griffin's thesis.

A *weak composition* of length n is a sequence of n non-negative integers. If α is a weak composition, we use α_i to denote its i^{th} entry. Given weak compositions α, β. we say γ is a *shuffle* of α and β if γ is obtained by inserting entries of α into β while preserving the relative order. A shuffle of more than two weak compositions can be defined similarly (or recursively). By convention, the unique shuffle of a single weak composition is just itself.

Let n be a positive number. The *staircase* of length n is the weak composition $\rho^{(n)} := (n-1, n-2, \ldots, 1, 0)$.[2] Take a partition λ with $1 \leqslant |\lambda| \leqslant n$ and a number $\ell(\lambda) \leqslant s < \infty$. We say a weak composition is a *(n, λ, s)-staircase* if it is a shuffle of the following weak compositions:

$$\rho^{\lambda'_1}, \cdots, \rho^{\lambda'_{\lambda_1}}, (s-1)^{n-|\lambda|},$$

where the last weak composition consists of $n - |\lambda|$ copies of $s-1$. Let $\mathcal{C}_{n,\lambda,s}$ be the set of weak compositions of length n that are entry-wise less than or equal to some (n, λ, s)-staircase. Finally, let $\mathcal{C}_{n,\lambda,\infty} := \bigcup_{s \geqslant \ell(\lambda)} \mathcal{C}_{n,\lambda,s}$.

Let α be a weak compositions of length n. We use x^α to denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then define $\mathcal{A}_{n,\lambda,s} := \{x^\alpha : \alpha \in \mathcal{C}_{n,\lambda,s}\}$. These monomials form Griffin's monomial basis.

Theorem 2.7. [Gri20, Theorem 3.1.17] *The quotient ring $R_{n,\lambda,s}$ has basis $\{m + I_{n,\lambda,s} : m \in \mathcal{A}_{n,\lambda,s}\}$.*

Remark 2.8. Consider (n, λ, s) where $|\lambda| > n$. Recall that we generalize $I_{n,\lambda,s}$ and $R_{n,\lambda,s}$ as $\mathbb{Q}[x_1, \cdots, x_n]$ and the trivial ring when $|\lambda| > n$. We may set $\mathcal{A}_{n,\lambda,s} = \mathcal{C}_{n,\lambda,s} = \varnothing$. Then Griffin's theorem still holds in this case: The trivial ring has the empty set as its basis. Again, our generalization is not introducing anything interesting, but will make our later work easier to state.

Notice that Griffin's monomial basis is closed under taking factor. This is a property shared by all standard monomial bases. It turns out that Griffin's basis is the standard monomial basis of $R_{n,\lambda,s}$ with respect to the graded reverse lexicographical order. Let $\mathcal{A}^C_{n,\lambda,s}$ denote the set of monomials of $x_1, \cdots, x_n$ that are not in $\mathcal{A}_{n,\lambda,s}$. We deduce the following:

Lemma 2.9. *Take (n, λ) with $|\lambda| \leqslant n$. If we can find a finite set of polynomials $G_{n,\lambda} \subseteq I_{n,\lambda,\infty}$ such that*

$$\langle \{\mathsf{in}_<(F) : F \in G_{n,\lambda}\} \rangle = \langle A^C_{n,\lambda,\infty} \rangle,$$

then $G_{n,\lambda}$ is a Gröbner basis of $I_{n,\lambda,\infty}$. Moreover, $A_{n,\lambda,\infty}$ is the set of standard monomials of $I_{n,\lambda,\infty}$.

Now take any s such that $\ell(\lambda) \leqslant s < \infty$. Then $G_{n,\lambda} \cup \{x_1^s, \cdots, x_n^s\}$ is a Gröbner basis of $I_{n,\lambda,s}$. Moreover, $A_{n,\lambda,s}$ is the set of standard monomials of $I_{n,\lambda,s}$.

Proof. The existence of $G_{n,\lambda}$ implies that $\langle \mathcal{A}^C_{n,\lambda,\infty} \rangle \subseteq \mathsf{in}_<(I_{n,\lambda,\infty})$. Thus, the set of standard monomials of $I_{n,\lambda,\infty}$ is a subset of $\mathcal{A}_{n,\lambda,\infty}$. Recall the standard monomials descends to a basis of $R_{n,\lambda,\infty}$. A subset of Griffin's monomial basis is a basis, so this "subset" has to be the whole set. In other

[2] Notice that our convention of a staircase is the reverse of the Griffin's convention.

words, monomials in $A_{n,\lambda,\infty}$ are all standard monomials. Since any monomial in $A_{n,\lambda,\infty}$ is not in $\mathsf{in}_<(I_{n,\lambda,\infty})$, we know $\langle \mathcal{A}^C_{n,\lambda,\infty}\rangle = \mathsf{in}_<(I_{n,\lambda,\infty})$. Thus, $G_{n,\lambda}$ is a Gröbner basis.

Now take $\ell(\lambda) < s < \infty$. We can write $\mathcal{A}^C_{n,\lambda,s}$ as a union of two sets:

$$\mathcal{A}^C_{n,\lambda,\infty} \quad \bigcup \quad \{\text{monomials divisible by some } x_i^s\}.$$

The first set is a subset of $\mathsf{in}_<(I_{n,\lambda,\infty})$, which is a subset of $\mathsf{in}_<(I_{n,\lambda,s})$. The second set of monomials is a subset of $\mathsf{in}_<(I_{n,\lambda,s})$ since $x_1^s, \cdots, x_n^s$ are generators of $I_{n,\lambda,s}$. Again, we have $\langle \mathcal{A}^C_{n,\lambda,s}\rangle \subseteq \mathsf{in}_<(I_{n,\lambda,s})$. We perform the argument in the previous paragraph. Consequently, we know $A_{n,\lambda,s}$ is the set of standard monomials of $I_{n,\lambda,s}$ and $\langle A^C_{n,\lambda,s}\rangle = \mathsf{in}_<(I_{n,\lambda,s})$. Finally, since $G_{n,\lambda} \subseteq I_{n,\lambda,\infty} \subseteq I_{n,\lambda,s}$, the set $G_{n,\lambda} \cup \{x_1^s, \cdots, x_n^s\}$ is a Gröbner basis of $I_{n,\lambda,s}$. $\square$

The rest of this paper aims to find the $G_{n,\lambda}$ in Lemma 2.9.

3. Finding generators of $\mathsf{in}_<(I_{n,\lambda,\infty})$ via container diagrams

We will find a finite set of monomials $B_{n,\lambda}$ that generates the same ideal as $A^C_{n,\lambda,\infty}$. First, we recall some combinatorics construction developed by Rhoades, Yu and Zhao [RYZ20].

3.1. Container diagram and coinversion code.

The *Young diagram* of a partition λ is a diagram consisting of boxes: row i of the diagram has a box at column $1, 2, \cdots, \lambda_i$. We use the convention that row 1 is the topmost row and column 1 is the leftmost column. Given n, λ, Rhoades, Yu and Zhao define the *container diagram*: This is the Young diagram of λ' together with numbers of $[n]$ filled. Each number can either be filled in a box of the Young diagram or just float somewhere above the Young diagram in a column. Numbers that are not in boxes are called the *floating numbers*. The filling should satisfy all of the following.

1. Every number in $[n]$ appears exactly once.
2. Every box can contain at most one number.
3. An empty box cannot have any number above it in its column.
4. Numbers are decreasing from top to bottom in each column.

Example 3.1. Let $n = 11$ and $\lambda = (3, 2, 2, 1)$. Then following would be a container diagram of (n, λ):

$$\sigma = \begin{array}{ccccc} & & 10 & & \\ 11 & & 7 & & 5 \\ \boxed{4} & \boxed{} & \boxed{6} & \boxed{9} & \\ \boxed{3} & \boxed{8} & \boxed{1} & & \\ \boxed{2} & & & & \end{array}$$

Remark 3.2. Our convention of the container diagram is slightly different from the convention in [RYZ20]. First, we allow empty boxes to appear in the container diagram. Second, we do not require $n \geqslant |\lambda|$.

Rhoades, Yu and Zhao use container diagrams to represent *ordered set partitions*, an ordered infinite sequence of pair-wise disjoint sets such that the union is $[n]$, where each part is allowed to be $\varnothing$. Naturally, each container diagram of (n, λ) corresponds to an ordered set partition of $[n]$. For instance, the diagram in the previous example represents:

$$(\{2, 3, 4, 11\}, \{8\}, \{1, 6, 7, 10\}, \{9\}, \{5\}, \varnothing, \cdots)$$

where all trailing sets are empty. For this reason, we denote the set of container diagrams for (n, λ) as $\mathcal{OP}_{n,\lambda}$.

Rhoades Yu and Zhao define the following map from $\mathcal{OP}_{n,\lambda}$ to $\mathbb{Z}^n_{\geqslant 0}$, the set of weak compositions with length n. Notice that they only define the map on container diagrams whose boxes are all filled. We can simply extend their definition by imagining ∞ is filled in all empty boxes.

Definition 3.3. [RYZ20] Let σ be an container diagram of (n, λ). We define $\mathsf{code}(\sigma)$ as a weak compositions of length n. Its i^{th} entry can be found as follows.

- Say the number i lives in a box in σ. Assume i is in row r. Consider the boxes that are in row r to the right of i or in row $r+1$ to the left of i. Then $\mathsf{code}(\sigma)_i$ is the number of boxes in this region that contain a number larger than i.
- Otherwise, i is floating. Assume i is in column c. Then $\mathsf{code}(\sigma)_i$ is $c-1$ plus the number of boxes in row 1 to the right of column c that contain a number larger than i.

Remember that when computing $\mathsf{code}(\sigma)$, we imagine all empty boxes of the Yound diagram contains ∞. We call $\mathsf{code}(\sigma)$ the *coinversion code* of σ.

If σ is the container diagram in Example 3.1, then

$$\mathsf{code}(\sigma) = (1, 0, 1, 3, 4, 2, 3, 0, 0, 2, 1).$$

It turns out that code is a bijection from $\mathcal{OP}_{n,\lambda}$ to $\mathbb{Z}^n_{\geqslant 0}$. An inverse of the map is given in the proof of [RYZ20, Theorem 3.6]. We briefly describe this inverse code^{-1}. Given weak composition α with length n, we compute $\sigma = \mathsf{code}^{-1}(\alpha) \in \mathcal{OP}_{n,\lambda}$ via the following algorithm. We start from the Young diagram of λ' and insert $1, 2, \ldots, n$ iteratively. Suppose we have inserted $1, \ldots, t-1$ and we want to insert t. For each column, t can only go to the bottom-most available spot within the column. We determine which column t should go to based on the value of α_t. First, arrange the positive integers $1, 2, 3, \cdots$ in the following way.

- Suppose column i has an empty box but column j does not. Then i comes before j.
- Suppose column i and column j have no empty boxes. Then i comes before j if $i < j$.
- Suppose column i and column j both have empty boxes. Say the bottom-most empty box in column i (resp. j) is at row r_i (resp. r_j). Then i comes before j if $r_i > r_j$ or $r_i = r_j$ and $j < i$.

Let $c_1, c_2, c_3, \cdots$ be the resulting rearrangement of $1, 2, 3, \cdots$. Readers may check that if we put t at column c_p, the t^{th} entry in the coinversion code of the resulting container diagram must be $p-1$. Thus, we insert t into column c_{α_t+1}. We obtain $\mathsf{code}^{-1}(\alpha)$ after inserting $1, 2, \ldots, n$.

Example 3.4. We present one step in computing $\mathsf{code}^{-1}((1, 0, 1, 3, 4, 2, 3, 0, 0, 2, 1))$ with $n = 11$ and $\lambda = (3, 2, 2, 1)$. After inserting 1 and 2 to the empty Young diagram of λ', we have

3		1	
2			

Now we insert 4. By the algorithm, we may arrange the positive numbers into $(c_1, c_2, \cdots) = (2, 4, 3, 1, 5, 6, 7, \cdots)$. Putting the number 4 at the bottom of column c_t will yield a container diagram whose coinversion code has $t+1$ as the 4^{th} entry. Since we want to have 3 as the 4^{th} entry, we put 4 at the bottom of column $c_{3+1} = 1$. After inserting all numbers in $[11]$, we obtain the container diagram in Example 3.1.

Remark 3.5. We make one simple observation about the insertion that will be used later. Suppose we are computing $\mathsf{code}^{-1}(\alpha)$ and we are about insert t. If $\alpha_t = 0$ and there is at least one empty box in the diagram, we will put t in a box.

One use of the container diagram and the coinversion code is to identify whether a weak composition lies in $\mathcal{C}_{n,\lambda,\infty}$. Rhoades, Yu and Zhao describe its image under $\mathsf{code}^{-1}(\cdot)$.

Theorem 3.6. [RYZ20, Theorem 3.6] *Consider (n, λ). The map $\mathsf{code}(\cdot)$ and $\mathsf{code}^{-1}(\cdot)$ restrict to a bijections between $\mathcal{C}_{n,\lambda,\infty} \subseteq \mathbb{Z}^n_{\geqslant 0}$ and*

$$\{\sigma \in \mathcal{OP}_{n,\lambda} : \sigma \textit{ has no empty boxes in the Young diagram}\} \subseteq \mathcal{OP}_{n,\lambda}.$$

Technically, Rhoades, Yu and Zhao only prove this theorem when $n \geqslant |\lambda|$. When $n < |\lambda|$, notice that every container diagram in $\mathcal{OP}_{n,\lambda}$ must have a empty box. Recall that we set $\mathcal{C}_{n,\lambda,\infty} = \varnothing$, so this theorem holds trivially.

3.2. **Finding a finite generating set for $\langle \mathcal{A}^C_{n,\lambda,\infty} \rangle$.** In this subsection, we use container diagrams to describe a set of monomials $\mathcal{B}_{n,\lambda}$ that can generate the same ideal as $\mathcal{A}^C_{n,\lambda,\infty}$. Notice that this process is immediate when $|\lambda| > n$, in which case $\mathcal{A}_{n,\lambda,\infty} = \varnothing$. We only need to worry about the case when $|\lambda| \leqslant n$. However, we need to make use of a recursive structure of $\mathcal{B}_{n,\lambda}$ later, so we choose to focus on (n, λ) such that $1 \leqslant |\lambda| \leqslant n+1$. Now, fix such (n, λ) throughout this subsection.

By Theorem 3.6, we know $\mathcal{A}^C_{n,\lambda,\infty}$ can be written as

$$\{x^{\mathsf{code}(\sigma)} : \sigma \in \mathcal{OP}_{n,\lambda,s} \text{ with an empty box}\}.$$

Let $\mathcal{C}^C_{n,\lambda,\infty}$ be the set of weak compositions that are not in $\mathcal{C}_{n,\lambda,\infty}$. We describe a set $\mathcal{D}_{n,\lambda} \subseteq \mathcal{C}^C_{n,\lambda,\infty}$, Then we check for any α in $\mathcal{C}^C_{n,\lambda,\infty}$, we can find $\beta \in \mathcal{D}_{n,\lambda}$ such that $\beta \leqslant_e \alpha$. Here, $\leqslant_e$ denotes entry-wise less than or equal to. Finally, we set

$$\mathcal{B}_{n,\lambda} := \{x^\beta : \beta \in \mathcal{D}_{n,\lambda}\}.$$

Clearly, $\mathcal{B}_{n,\lambda}$ generates the same ideal as $\mathcal{A}^C_{n,\lambda,\infty}$. Now, we define $\mathcal{D}_{n,\lambda}$ and check it has the desired properties.

Definition 3.7. Let $\mathcal{D}_{n,\lambda}$ be the set of $\mathsf{code}(\sigma)$ where σ ranges over elements in $\mathcal{OP}_{n,\lambda}$ satisfying all of the following three conditions.

- Condition 1: The Young diagram in σ has exactly one empty box.
- Condition 2: The filling of the Young diagram is decreasing in each row from left to right. Moreover, the empty box in the Young diagram is at row 1 column 1.
- Condition 3: If a number i is floating, then every column on its left has a box that is empty of contains a number larger than i.

Example 3.8. Consider the (n, λ) and the container diagram σ in Example 3.1. Clearly, σ satisfies condition 1. However, σ does not satisfy condition 2, since the only box is not at column 1. Moreover, the rows are not decreasing from left to right. It does not satisfy condition 3 either, since 5 is floating but the 4 in column 1 row 1 is less than 5. Here is container diagram γ that satisfies all three conditions for the same (n, λ):

$$\gamma = \begin{array}{cccc} & 11 & & \\ & 10 & 7 & 5 \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{4} \\ \hline \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{1} & \\ \cline{1-3} \multicolumn{1}{|c|}{2} & & & \\ \cline{1-1} \end{array}$$

Thus, $\mathsf{code}(\gamma) = (1, 0, 0, 1, 3, 1, 2, 0, 0, 1, 1)$ is in $\mathcal{D}_{n,\lambda}$.

Then we check $\mathcal{D}_{n,\lambda}$ has the desired property.

Proposition 3.9. *For any $\alpha \in \mathcal{C}^C_{n,\lambda,\infty}$, we can find $\beta \in \mathcal{D}_{n,\lambda}$ with $\beta \leqslant_e \alpha$. Thus, $\langle B_{n,\lambda} \rangle = \langle A^C_{n,\lambda,\infty} \rangle$.*

Example 3.10. Consider the (n, λ) and the container diagram σ in Example 3.1. Let $\alpha = \mathsf{code}(\sigma)$. Since σ contains an empty box, we know $\alpha \in \mathcal{C}^C_{n,\lambda,\infty}$. Then β can be $\mathsf{code}(\gamma)$ where γ is the container diagram in Example 3.8. We have $\beta \in \mathcal{D}_{n,\lambda}$ and $\beta \leqslant_e \alpha$.

Remark 3.11. It is important to note that $\mathcal{D}_{n,\lambda}$ is not the smallest subset that satisfies Proposition 3.9. In other words, two distinct elements α and β in $\mathcal{D}_{n,\lambda}$ may satisfy $\alpha \leqslant_e \beta$. Readers may check when $n = 4$, $\lambda = (3, 1)$, $\mathcal{D}_{n,\lambda}$ contains both $(0, 0, 0, 1)$ and $(1, 0, 0, 1)$. Thus, if our Gröbner basis contains a polynomial of leading monomial x^α for each $\alpha \in \mathcal{D}_{n,\lambda}$, it cannot be the reduced Gröbner basis. It would be a good problem to find a new definition of $\mathcal{D}_{n,\lambda}$ such that Proposition 3.9 is satisfied and $\mathcal{D}_{n,\lambda}$ is minimal.

Now we prove Proposition 3.9. The idea is to turn the proposition into an equivalent statement regarding $\mathcal{OP}_{n,\lambda}$.

Proof of Proposition 3.9. Recall that $\mathsf{code}(\cdot)$ is a bijection from $\mathcal{OP}_{n,\lambda}$ to $\mathbb{Z}^n_{\geqslant 0}$. By Theorem 3.6, it is enough to show the following claim:
Claim: For any $\sigma \in \mathcal{OP}_{n,\lambda}$ with at least one empty box, we can find $\gamma \in \mathcal{OP}_{n,\lambda}$ satisfying the three conditions in the Definition 3.7 such that $\mathsf{code}(\gamma) \leqslant_e \mathsf{code}(\sigma)$.

Our approach to show the claim is straightforward, consisting of the following three lemmas. It is clear that these three lemmas would imply our claim, hence finishing the proof. □

Lemma 3.12. *For any $\sigma \in \mathcal{OP}_{n,\lambda}$ with at least one empty box, we can find $\gamma \in \mathcal{OP}_{n,\lambda}$ satisfying Condition 1 such that $\mathsf{code}(\gamma) \leqslant_e \mathsf{code}(\sigma)$.*

Lemma 3.13. *For any $\sigma \in \mathcal{OP}_{n,\lambda}$ satisfying condition 1, we can find $\gamma \in \mathcal{OP}_{n,\lambda}$ satisfying condition 1 and 2 such that $\mathsf{code}(\gamma) \leqslant_e \mathsf{code}(\sigma)$.*

Lemma 3.14. *For any $\sigma \in \mathcal{OP}_{n,\lambda}$ satisfying condition 1 and 2, we can find $\gamma \in \mathcal{OP}_{n,\lambda}$ satisfying condition 1, 2 and 3 such that $\mathsf{code}(\gamma) \leqslant_e \mathsf{code}(\sigma)$.*

Now we prove these three Lemmas. The proofs are all constructive: we give an algorithm to construct the γ from σ in each proof.

Proof of Lemma 3.12. If σ has exactly one empty box, we are done by setting $\gamma = \sigma$. Now assume σ has more than one box. Let $\alpha = \mathsf{code}(\sigma)$ and consider the insertion algorithm that computes $\mathsf{code}^{-1}(\alpha)$. We focus on the quantity:

$$\text{number of empty boxes} \quad - \quad \text{number of elements in } [n] \text{ that have not been inserted.}$$

Recall that we assume $|\lambda| \leqslant n + 1$ throughout this subsection. Thus, this quantity was $|\lambda| - n \leqslant 1$ when the algorithms starts, and becomes larger than 1 when the algorithm ends. Moreover, each iteration can only increase this quantity by 0 or 1. Thus, we can find $N < n$ such that right before inserting N, this quantity is 1. At this moment, there are $n - N + 1$ numbers waiting to be inserted. Thus, there are $n - N + 2$ empty boxes in diagram. We obtain weak composition β from α by setting $\beta_i = \alpha$ if $i < N$ and $\beta_i = 0$ if $i \geqslant N$. Clearly, $\beta \leqslant_e \alpha$. When we compute $\mathsf{code}^{-1}(\beta)$, the first $N - 1$ iterations behave the same as computing $\mathsf{code}^{-1}(\alpha)$. By Remark 3.5, the algorithm will insert $N, N + 1, \cdots, n$ into boxes, leaving exactly one empty box at the end. Thus, $\mathsf{code}^{-1}(\beta)$ is the γ we want. □

Proof of Lemma 3.13. In this proof, we imagine the empty spaces of a container diagram are filled with ∞. Define the row inversion of a container diagram to be a pair of boxes on the same row with the box on the left containing a larger number than the box on the right. With this convention and definition, condition 2 can be rephrased as: "There are no row inversions."

Now suppose $\sigma \in \mathcal{OP}_{n,\lambda}$ satisfies condition 1 and has at least one row inversion. We just need to find $\sigma' \in \mathcal{OP}_{n,\lambda}$ satisfying condition 1 and having less row inversions than σ. Moreover, we need to make sure $\mathsf{code}(\sigma') \leqslant_e \mathsf{code}(\sigma)$.

Look at the bottom-most row of boxes in σ where a row inversion appears. We can find two adjacent boxes in this row containing l_0, r_0 with $l_0 < r_0$ and l_0 is on the left. (The name l stands for "left" and r stands for "right"). Let $l_1 < l_2 < \cdots < l_m$ be the numbers in boxes above l_0. Let

l_{-1} be the number immediately underneath l_0, if it exists. Define $r_{-1}, r_1, r_2, \cdots$ in the same way. These two columns of the boxes in σ looks like:

$$\begin{array}{cc}
\hline
\multicolumn{1}{|c|}{l_m} & \multicolumn{1}{c|}{r_m} \\
\hline
\vdots & \vdots \\
\hline
\multicolumn{1}{|c|}{l_1} & \multicolumn{1}{c|}{r_1} \\
\hline
\multicolumn{1}{|c|}{l_0} & \multicolumn{1}{c|}{r_0} \\
\hline
\multicolumn{1}{|c|}{l_{-1}} & \multicolumn{1}{c|}{r_{-1}} \\
\hline
\vdots & \vdots
\end{array}$$

Now we obtain σ' by doing the following operations to σ. Find the largest $i \in [m]$ such that $l_j < r_j$ for $j = 0, 1, \ldots, i$. Then for $j = 0, 1, \ldots, i$, we swap l_j and r_j. If $i = m$, we also look at the numbers that were floating above l_m in σ. We find the floating numbers that are less than r_m and move them to float in the column on the right.

First, we check σ' is a valid container diagram. We just need to make sure the columns are strictly decreasing from top to bottom.

- If l_{-1} exists, we check it is smaller than the number above it in σ', which is r_0: $r_0 > l_0 > l_{-1}$.
- If r_{-1} exists, since we pick the lowest row in σ with a row inversion, we know $l_0 > l_{-1} > r_{-1}$. Thus, r_{-1} is smaller than the number above it in σ'.
- If $i < m$, we need to check r_i and l_i are smaller than the numbers above them in σ': $l_{i+1} > r_{i+1} > r_i$ by the maximality of i and $r_{i+1} > r_i > l_i$.
- If $i = m$, we need to check r_m and l_m are smaller than the numbers floating above them in σ'. If a number is floating above r_m, by our construction, it is larger than r_m. If a number is floating above l_m, there are two possibilities: it was floating above r_m or l_m in σ. In either case, we know it is larger than l_m.

Finally, σ' clearly has less row inversions than σ. It remains to check $\mathsf{code}(\sigma') \leqslant_e \mathsf{code}(\sigma)$. A routine analysis would yield: $\mathsf{code}(\sigma')_j = \mathsf{code}(\sigma)_j - 1$ if $j = l_0, \cdots, l_i$, and $\mathsf{code}(\sigma')_j = \mathsf{code}(\sigma)_j$ otherwise. ☐

Proof of Lemma 3.13. Suppose the floating number i in σ does not satisfy condition 3. In other words, a column on the left of j does not have boxes that are empty or containing a number larger than i. We simply ask i to float in that column. This operation clearly reduces the i^{th} entry of the coinversion code and preserves condition 1 and 2. We may eliminate all violations of condition 3 by repeatedly applying this operation. ☐

3.3. **Recursive description of $\mathcal{D}_{n,\lambda}$.** We end this section by observing a recursive structure of $\mathcal{D}_{n,\lambda}$. We need the following notation from Griffin's thesis:

Definition 3.15. Let λ be a partition. We define $\lambda^{(j)}$ as the partition such that the conjugate of $\lambda^{(j)}$ is obtained from λ' by decreasing its j^{th} entry by 1. By convention, we let $\lambda^{(0)} = \lambda$.

Notice that $\lambda^{(j)}$ is not well-defined for all j. Clearly, $\lambda^{(j)}$ is well-defined if and only if the rightmost cell in row j of the Young diagram of λ' is the bottommost cell in its column.

Example 3.16. Let $\lambda = (3, 2)$. Then

$$\lambda^{(0)} = (3, 2), \quad \lambda^{(2)} = (3, 1), \quad \lambda^{(3)} = (2, 2),$$

but $\lambda^{(1)}$ is not defined.

Proposition 3.17. *If $1 \leqslant |\lambda| \leqslant n + 1$ and $n > 1$, we can write $\mathcal{D}_{n,\lambda}$ recursively as*

$$\mathcal{D}_{n,\lambda} = \bigsqcup_j (\lambda'_{j+1} \circ \mathcal{D}_{n-1,\lambda^{(j)}}),$$

where j ranges over j such that $\lambda^{(j)}$ is defined, but j cannot be 0 if $|\lambda| = n+1$. Here, $\lambda'_{j+1}\circ$ is the operator that prepends λ'_{j+1} to a weak composition.

For the base case, we have $D_{1,(1)} = \{(1)\}$ and $D_{1,(2)} = D_{1,(1,1)} = \{(0)\}$

Proof. First, assume $n > 1$. Recall that $\mathcal{D}_{n,\lambda}$ is the set of coinversion codes of $\sigma \in \mathcal{OP}_{n,\lambda}$ where σ satisfies the three conditions in Definition 3.7. Consider such a σ.

Suppose 1 is not floating in σ, by condition 2, there are no boxes to its right or under it. Thus, 1 is in the rightmost box of row j for some j that $\lambda^{(j)}$ is defined. Let σ' be the container diagram obtained by ignore this box and decrease all other numbers by 1. Clearly, σ' is in $\mathcal{OP}_{n,\lambda^{(j)}}$ and satisfies the three conditions. In addition,

$$\mathsf{code}(\sigma) = \mathsf{code}(\sigma)_1 \circ \mathsf{code}(\sigma').$$

Since 1 is on the right end of row j, we know $\mathsf{code}(\sigma)_1$ is just the number of boxes in row $j+1$ of σ, which is λ'_{j+1}.

Now suppose 1 is floating in σ. This can happen only when $|\lambda| \leqslant n$. Condition 3 forces the 1 to float in column $\ell(\lambda)+1$. Similarly, we may let σ' be the container diagram obtained by ignore 1 in σ and decrease all other numbers by 1. Then $\sigma' \in \mathcal{OP}_{n,\lambda^{(0)}}$ satisfying the three conditions and $\mathsf{code}(\sigma) = \mathsf{code}(\sigma)_1 \circ \mathsf{code}(\sigma')$. We see $\mathsf{code}(\sigma)_1 = \ell(\lambda) = \lambda'_1$, which proves the recurrence relation. The base cases are immediate. ☐

4. Recursive construction of the Gröbner basis

Consider n and λ with $1 \leqslant |\lambda| \leqslant n+1$. In this section, we construct $G_{n,\lambda} \subseteq I_{n,\lambda,\infty}$ such that $\{\mathsf{in}_<(F) : F \in G_{n,\lambda}\} = B_{n,\lambda}$. Define $J_{n,\lambda}$ as an ideal of $\mathbb{Z}[x_1, \cdots, x_n]$ with the same generators as $I_{n,\lambda,\infty}$ in Definition 2.2. In fact, our $G_{n,\lambda}$ will be a subset of $J_{n,\lambda}$. When $n = 1$, we simply let $G_{n,(1)} = \{e_1([1])\} = \{x_1\}$ and $G_{n,(2)} = G_{n,(1,1)} = \{e_0([1])\} = \{1\}$. For $n > 1$, our recursive approach can be summarized as follows:

Remark 4.1. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a weak composition of length n. Let J be an ideal of $\mathbb{Z}[x_2, \cdots, x_n]$ generated by $f_1, \ldots, f_m$. Suppose we have a polynomial $f \in J$ with $\mathsf{in}_<(f) = x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. We may write f as $f_1 g_1 + \cdots + f_n g_n$ for some $g_1, \cdots, g_n \in \mathbb{Z}[x_2, \cdots, x_n]$.

Now consider homogeneous polynomials $F_1, \cdots, F_m \in \mathbb{Z}[x_1, \cdots, x_n]$ such that each $F_i - x_1^{\alpha_1} f_i$ is divisible by $x_1^{\alpha_1+1}$. In other words, $F_i = x_1^{\alpha_1} f_i + x_1^{\alpha_1+1}\tilde{F}_i$ for some polynomial $\tilde{F}_i$. Then $F = F_1 g_1 + \cdots + F_m g_m$ is a homogeneous polynomial. Moreover,

$$F = \sum_{i=1}^n F_i g_i = \sum_{i=1}^n x_1^{\alpha_1} f_i g_i + \sum_{i=1}^n x_1^{\alpha_1+1}\tilde{F}_i g_i = x_1^{\alpha_1}\sum_{i=1}^n f_i g_i + x_1^{\alpha_1+1}\sum_{i=1}^n \tilde{F}_i g_i = x_1^{\alpha_1} f + x_1^{\alpha_1+1}\sum_{i=1}^n \tilde{F}_i g_i.$$

Thus, $\mathsf{in}_<(F) = x^{\alpha_1}\mathsf{in}_<(f) = x^{\alpha}$.

If the leading monomial in f has coefficient 1, then so is the leading monomial in F.

Now fix $n > 1$ and $1 \leqslant |\lambda| \leqslant n+1$. Take any j such that $\lambda^{(j)}$ is defined. Let $J'_{n,\lambda^{(j)}}$ be the ideal of $\mathbb{Z}[x_2 \ldots, x_n]$ obtained from $J_{n-1,\lambda^{(j)}}$ by shifting each x_i to x_{i+1}. In other words, $J'_{n,\lambda^{(j)}}$ is generated by $e_d(S)$ where $S \subseteq \{2, \cdots, n\}$ and $d > |S| - p^{n-1}_{|S|}(\lambda^{(j)})$. Set $a = \lambda'_{j+1}$. Remark 4.1 suggests that for each such $e_d(S)$, we should find a homogeneous polynomial $F_{d,S} \in J_{n,\lambda}$ such that $F_{d,S} - e_d(S)$ is divisible by x_1^{a+1}:

Lemma 4.2. *For each $e_d(S)$ among the generators $J'_{n,\lambda^{(j)}}$, we let*

$$F_{d,S} := \begin{cases} x_1^a e_d(S \sqcup \{1\}) & \text{if } |S| \geqslant n-j, \\ x_1^a e_d(S) & \text{if } |S| < n-j. \end{cases}$$

Then $F_{d,S}$ is a homogeneous polynomial in $J_{n,\lambda}$ such that $F_{d,S} - e_d(S)$ is divisible by x_1^{a+1}

Our proof relies on (1) and uses arguments similar to Griffin's arguments in section 3 of [Gri20].

Proof. Clearly, $F_{d,S}$ is homogeneous. Then observe that

$$p_m^{n-1}(\lambda^{(j)}) = p_{m+1}^n(\lambda^{(j)}) = \begin{cases} p_{m+1}^n(\lambda) - 1 & \text{if } n-m \leqslant j, \\ p_{m+1}^n(\lambda) & \text{if } n-m > j. \end{cases}$$

If $|S| \geqslant n-j$, we know $d > |S| - p_{|S|}^{n-1}(\lambda^{(j)}) = (|S|+1) - p_{|S|+1}^n(\lambda)$. Thus, $d+j$ $e_d(S \sqcup \{1\})$ is in $J_{n,\lambda}$, so is $F_{d,S}$. By (1),

$$F_{d,S} = x_1^a e_d(S) + x_1^{a+1} e_{d-1}(S).$$

Thus, $F_{d,S} - x_1^a e_d(S)$ is divisible by x_1^{a+1}.

If $|S| < n-j$, we clearly know $F_{d,S} - x_1^a e_d(S)$ is divisible by x_1^{a+1}. It remains to check $F_{d,S} \in J_{n,\lambda}$. By repeatedly applying (1), we have

$$F_{d,S} = x_1^a e_d(S) = \sum_{k=1}^{a} (-1)^{k-1} x_1^{a-k} e_{d+k}(S \sqcup \{1\}) + (-1)^a e_{d+a}(S).$$

Then we can show each term on the right hand side is in $J_{n,\lambda}$. We know $d > |S| - p_{|S|}^{n-1}(\lambda^{(j)}) = |S| - p_{|S|+1}^n(\lambda)$. Then for $k \geqslant 1$, $d+k > (|S|+1) - p_{|S|+1}^n(\lambda)$, so $e_{d+k}(S \sqcup \{1\})$ is in $J_{n,\lambda}$. Recall that $a = \lambda'_{j+1}$. The assumption $|S| < n-j$ implies $j+1 \leqslant n-|S|$, so $a = \lambda'_{j+1} \geqslant \lambda'_{n-|S|}$. We have

$$d + a > |S| - p_{|S|+1}^n(\lambda) + a \geqslant |S| - p_{|S|+1}^n(\lambda) + \lambda'_{n-|S|} = |S| - p_{|S|}^n(\lambda),$$

so $e_{d+a}(S)$ is in $J_{n,\lambda}$. □

Now take $\alpha \in D_{n,\lambda}$. By Proposition 3.9, $\alpha_1 = \lambda'_{j+1}$ for some j such that $\lambda^{(j)}$ is defined. Moreover, $(\alpha_2, \cdots, \alpha_n) \in D_{n,\lambda^{(j)}}$. By recursion, we let f be the polynomial in $G_{n-1,\lambda^{(j)}}$ with $\mathsf{in}_<(f) = x^{(\alpha_2,\cdots,\alpha_n)}$. Obtain $\tilde{f}$ from f by shifting all x_i to x_{i+1}. We know $\tilde{f} \in J_{n,\lambda^{(j)}}$, so we may write $\tilde{f}$ as $\sum_{d,S} g_{d,S} e_d(S)$, where $g_{d,S} \in \mathbb{Z}[x_2, \cdots, x_n]$. By Remark 4.1 and Lemma 4.2, if we let $F := \sum_{d,S} g_{d,S} F_{d,S}$, then F is a homogeneous polynomial in $J_{n,\lambda}$ with $\mathsf{in}_<(F) = x^\alpha$ and leading coefficient 1.

Theorem 4.3. *The $G_{n,\lambda}$ we construct is a Gröbner basis of $I_{n,\lambda,\infty}$. Its polynomials have integer coefficients with leading coefficient* 1.

Proof. By the recursive construction, each polynomial in $G_{n,\lambda}$ has integer coefficients and leading coefficient 1. We know $G_{n,\lambda} \subseteq J_{n,\lambda}$, so $G_{n,\lambda} \subset I_{n,\lambda,\infty}$. We know $\{\mathsf{in}_<(F) : F \in G_{n,\lambda}\} = \{B_{n,\lambda}\}$. By Proposition 3.9, $\langle \{\mathsf{in}_<(F) : F \in G_{n,\lambda}\} \rangle = \langle A_{n,\lambda,\infty}^C \rangle$. Then the proof is finished by Lemma 2.9. □

Example 4.4. Suppose $n = 5$ and $\lambda = (2,2,1)$. We would like to construct $F \in J_{5,(2,2,1)}$ with leading monomial $x^{(0,1,2,0,0)}$. Here, the weak composition $(0,1,2,0,0) \in D_{5,(2,2,1)}$ comes from the following element of $\mathcal{OP}_{5,(2,2,1)}$:

$$\begin{array}{ccc} & & 3 \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{1} & \\ \cline{1-2} \end{array}$$

We construct F recursively. To make our computation clear, we reverse the recursive process.

- We can find
$$e_0(\{1\}) \in J_{1,(2)}$$
with leading monomial $x^{(0,0)}$.

- Based on the previous polynomial, we can find
$$e_0(\{1,2\}) \in J_{2,(2,1)}$$
with leading monomial $x^{(0,0)}$.
- Based on the previous polynomial, we can find
$$x_1^2 e_0(\{2,3\} \in J_{3,(2,1)}$$
with leading monomial $x^{(2,0,0)}$. It can be written as
$$x_1 e_1(\{1,2,3\}) - e_2(\{1,2,3\}) + e_2(\{2,3\}).$$
- Based on the previous polynomial, we can find
$$x_1 x_2 e_1(\{1,2,3,4\}) - x_1 e_2(\{1,2,3,4\}) + x_1 e_2(\{3,4\}) \in J_{4,(2,1,1)}$$
with leading monomial $x^{(1,2,0,0)}$. It can be written as
$$x_1 x_2 e_1(\{1,2,3,4\}) - x_1 e_2(\{1,2,3,4\}) + e_3(\{1,3,4\}).$$
- Based on the previous polynomial, we can find
$$x_2 x_3 e_1(\{1,2,3,4,5\}) - x_2 e_2(\{1,2,3,4,5\}) + e_3(\{1,2,4,5\}) \in J_{5,(2,2,1)}$$
with leading monomial $x^{(0,1,2,0,0)}$.

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