CONNECTION BETWEEN SCHUBERT POLYNOMIALS AND TOP LASCOUX POLYNOMIALS

TIANYI YU

Abstract. Schubert polynomials form a basis of the polynomial $\mathbb{Q}[x_1, x_2, \cdots]$. This basis and its structure constants have received extensive study. Recently, Pan and Yu initiated the study of top Lascoux polynomials. These polynomials form a basis of the vector space $\widehat{V}$, a subalgebra of $\mathbb{Q}[x_1, x_2, \cdots]$ where each graded piece has finite dimension. This paper connects Schubert polynomials and top Lascoux polynomials via a simple operator. We use this connection to show these two bases share the same structure constants. We also translate several results on Schubert polynomials to top Lascoux polynomials, including combinatorial formulas for their monomial expansions and supports.

1. Introduction

For a permutation w, Lascoux and Schützenberger [LS82a] recursively define the *Schubert polynomial* $\mathfrak{S}_w$ using *divided difference operators*. These polynomials represent Schubert cycles in flag varieties and have been extensively investigated from various perspectives. We summarize some significant results on Schubert polynomials relevant to this paper.

(1) The set of all Schubert polynomials forms a basis of the polynomial ring $\mathbb{Q}[x_1, x_2, \cdots]$. Products of Schubert polynomials can be expanded positively into Schubert polynomials (i.e. the expansion only involves positive integer coefficients):

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w,$$

The coefficient $c_{u,v}^w$ is known as the *Schubert structure constant*. A major open problem in algebraic combinatorics is to compute $c_{u,v}^w$ combinatorially.

(2) Lam, Lee and Shimozono [LLS21] introduced the *(reduced) bumpless pipedreams (BPD)* to compute the monomial expansion of Schubert polynomials.

(3) Adve, Robichaux, and Yong [ARY21] introduced *perfect tableaux* to compute the support of Schubert polynomials.

(4) The Schubert polynomials have the *saturated Newton polytope (SNP)* property [FMSD18].

(5) The Schubert polynomial can be expanded positively into *key polynomials* [RS95].

The key polynomials mentioned above are denoted as κ_α, where α is a *weak composition*. They are the characters of Demazure modules [Dem74]. Lascoux [Las03] introduced an inhomogeneous analogue of κ_α known as the *Lascoux polynomial* $\mathfrak{L}_\alpha$. The lowest-degree terms of $\mathfrak{L}_\alpha$ form κ_α. Recently, Pan and Yu [PY23] introduced the *top Lascoux polynomial* $\widehat{\mathfrak{L}}_\alpha$ which consists of the highest-degree terms of $\mathfrak{L}_\alpha$. Let $\widehat{V}$ be the $\mathbb{Q}$-span of all top Lascoux polynomials. Unlike the Schubert polynomials, the set of all top Lascoux polynomials is not linearly independent. To resolve this, Pan and Yu called a weak composition *snowy* if its positive entries are distinct. Then $\{\widehat{\mathfrak{L}}_\alpha : \alpha \text{ is snowy}\}$ forms a basis of $\widehat{V}$. By [PY23, Theorem 1.2], every top Lascoux polynomial is a scalar multiple of a top Lascoux indexed by snowy weak composition. In the rest of this paper, we only focus on $\widehat{\mathfrak{L}}_\alpha$ when α is snowy.

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Pan and Yu showed that $\widehat{V}$ is closed under multiplication. Then $\widehat{V}$ can be viewed as a graded algebra where the grading is given by degrees of polynomials. Pan and Yu computed the Hilbert series of $\widehat{V}$. Intuitively, $\widehat{V}$ is much smaller than the polynomial ring $\mathbb{Q}[x_1, x_2, \cdots]$ which has no Hilbert series.

Just like the Schubert polynomials, $\widehat{\mathfrak{L}}_\alpha$ can be defined recursively using divided difference operators (see (1)). This resemblance leads to a strong connection between Schubert polynomials and top Lascoux polynomials, which is the main focus of this paper.

Definition 1.1. Define the following involution on polynomials in $\mathbb{Q}[x_1, \cdots, x_n]$ where each variable has degree at most m:

$$r_{m,n}(f) := (x_1 \cdots x_n)^m f(x_n^{-1}, \cdots, x_1^{-1}).$$

In §3, we show that each top Lascoux polynomial can be realized as $r_{m,n}(\mathfrak{S}_w)$ for some m, n, w and vice versa. Following this connection, we translate the results on $\mathfrak{S}_w$ summarized above to $\widehat{\mathfrak{L}}_\alpha$.

(1) Products of top Lascoux polynomials can be expanded positively into top Lascoux polynomials:

$$\widehat{\mathfrak{L}}_\alpha \widehat{\mathfrak{L}}_\gamma = \sum_\sigma d^\sigma_{\alpha,\gamma} \widehat{\mathfrak{L}}_\sigma.$$

We call the coefficient *top Lascoux structure constants*. Every $d^\sigma_{\alpha,\gamma}$ is $c^w_{u,v}$ for some permutations u, v, w and vice versa (see §4).

(2) We give a monomial expansion of top Lascoux polynomials using (modified) bumpless-pipedreams (see §5).

(3) The support of top Lascoux polynomials can be computed using perfect tableaux (see §6).

(4) The top Lascoux polynomials have the SNP property (See §6).

(5) The top Lascoux polynomials can be expanded positively into key polynomials (see §7).

Besides helping us understand $\widehat{\mathfrak{L}}_\alpha$, our results may also shed light on the study of Schubert polynomials. By (1), computing the Schubert structure constants is the same as computing the top Lascoux structure constants, which happens in the algebra $\widehat{V}$. In other words, we move this problem into an algebra where each graded piece has finite dimension.

We conclude our introduction with one potential application of our results on $\widehat{\mathfrak{L}}_\alpha$. Schubert polynomials have an inhomogeneous analogue known as the *Grothendieck polynomials* $\mathfrak{G}_w$ [LS82b]. Their lowest-degree terms of $\mathfrak{G}_w$ form the Schubert polynomial $\mathfrak{S}_w$. Recently, their top-degree components $\widehat{\mathfrak{G}}_w$, which we call *top Grothendieck polynomials*, have attracted increasing attention:

- The difference of degrees between $\widehat{\mathfrak{G}}_w$ and $\mathfrak{S}_w$ yields the Castelnuovo-Mumford regularity of matrix Schubert varieties [KM01][RRR+21]. Pechenik, Speyer and Weigandt gave a combinatorial formula to compute the degree of $\widehat{\mathfrak{G}}$ [PSW21].
- Mészáros, Setiabrata, and St Dizier conjectured that the support of Grothendieck polynomials can be characterized using the support of top Grothendieck polynomials [MSSD22].

Reiner and Yong conjectured a positive formula for the expansion of $\mathfrak{G}_w$ into the $\mathfrak{L}_\alpha$ [RY21]. This conjecture was proved by Shimozono and Yu [SY21]. Thus, $\widehat{\mathfrak{G}}_w$ expands positively into the $\widehat{\mathfrak{L}}_\alpha$. Our combinatorial formulas of the monomial expansion and support of $\widehat{\mathfrak{L}}_\alpha$ might lead to formulas for $\widehat{\mathfrak{G}}$.

The rest of the paper is structured as follows. In Section §2, we provide an overview of the necessary background information. In §3, we use $r_{m,n}$ to relate the Schubert polynomials and top Lascoux polynomials. The subsequent sections explore various applications of this relationship. In Section §4, we examine the connection between the structure coefficients of top Lascoux polynomials and Schubert polynomials. In §5, we derive a combinatorial formula for top Lascoux polynomials from the BPD formula of Schubert polynomials. In §6, we analyze the support of top Lascoux polynomials by utilizing the support of Schubert polynomials. Finally, in §7, we translate the key expansion of Schubert polynomials to obtain a key expansion of top Lascoux polynomials.

2. Background

2.1. Schubert polynomials. Let S_+ be the group of permutations of $\{1, 2, \cdots\}$ where only finitely many elements are permuted. The simple transpositions $s_1, s_2, \ldots$ where $s_i = (i, i+1)$ generate S_+. For any positive number n, S_n is a subgroup of S_+ consisting of w that only permutes $[n] = \{1, 2, \cdots, n\}$. We represent $w \in S_+$ by its *one-line notation* $[w(1), \ldots, w(n)]$ for some n large enough such that $w \in S_n$.

A *weak composition* $\alpha = (\alpha_1, \alpha_2, \ldots,)$ is an infinite sequence of non-negative numbers with finitely many positive entries. The *support* of α is the set $\mathsf{supp}(\alpha) := \{i : \alpha_i > 0\}$. We represent α as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\mathsf{supp}(\alpha) \subseteq [n]$. Let $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ and $|\alpha| := \sum_{i \geqslant 1} \alpha_i$.

We say (i, j) is an inversion of $w \in S_+$ if $i < j$ and $w(i) > w(j)$. The *inversion code* of w, denoted as $\mathsf{invcode}(w)$ is a weak composition defined as

$$\mathsf{invcode}(w)_i := |\{j : (i, j) \text{ is an inversion of } w\}|.$$

The *Schubert polynomials* $\mathfrak{S}_w$ are indexed by permutations from S_+. When a weak composition is weakly decreasing, we say it is a *partition*. When $\mathsf{invcode}(w)$ is a partition, we say w is a *dominant permutation*. Define the *Newton divided difference operator*:

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}},$$

where $s_i f$ is the operator that swaps x_i and x_{i+1}. Now we can define the *Schubert polynomial* of $w \in S_+$ recursively [LS82a].

$$\mathfrak{S}_w = \begin{cases} x^{\mathsf{invcode}(w)} & \text{if } w \text{ is dominant} \\ \partial_i(\mathfrak{S}_{ws_i}) & \text{if } w(i) < w(i+1). \end{cases}$$

The set of Schubert polynomials form a $\mathbb{Q}$-basis of the polynomial ring $\mathbb{Q}[x_1, x_2, \cdots]$. For $u, v \in S_+$, the product $\mathfrak{S}_u \mathfrak{S}_v$ can be expanded into Schubert polynomials. Let $c_{u,v}^w$ be the coefficient of $\mathfrak{S}_w$ in this expansion. By geometric results, $c_{u,v}^w$ is a non-negative integer known as the *Schubert structure constants*.

2.2. Key polynomials and top Lascoux polynomials. The key polynomials κ_α are indexed by weak compositions. Lascoux and Schützenberger [LS82b] define the key polynomials recursively. using the operator $\pi_i(f) := \partial_i(x_i f)$:

$$\kappa_\alpha := \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition,} \\ \pi_i(\kappa_{s_i \alpha}) & \text{if } \alpha_i < \alpha_{i+1}, \end{cases}$$

where s_i swaps the i^{th} and $(i+1)^{\text{th}}$ entries of α.

The *top Lascoux polynomial* $\widehat{\mathfrak{L}}_\alpha$ are homogeneous polynomials indexed by *snowy* weak compositions: weak compositions whose positive entries are distinct. Following [PY23, Lemma 4.23], we may define these polynomials recursively. Define the operator $\widehat{\pi}_i$ as

$$\widehat{\pi}_i(f) := \pi_i(x_{i+1} f) = x_i x_{i+1} \partial_i(f).$$

Then define

$$\widehat{\mathfrak{L}}_\alpha := \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition,} \\ \widehat{\pi}_i(\widehat{\mathfrak{L}}_{s_i \alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases} \tag{1}$$

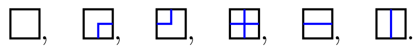
By the study of Pan and Yu, the vector space

$$\widehat{V} := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{L}}_\alpha : \alpha \text{ is a snowy weak composition}\} \tag{2}$$

is an sub-algebra of $\mathbb{Q}[x_1, x_2, \dots]$. Its basis is given by the spanning set in (2). and its Hilbert series is $\prod_{m>0}\left(1 + \frac{q^m}{1-q}\right)$. In particular, each graded piece of $\widehat{V}$ has finite dimension.

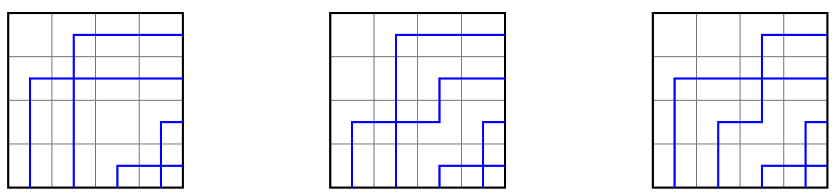
For weak compositions α, γ, and δ, let $d_{\alpha,\gamma}^{\delta}$ be the coefficient of $\widehat{\mathfrak{L}}_\delta$ in the expansion of $\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma$. We call them the *top Lascoux structure constants*. Later in §4, we show each $d_{\alpha,\gamma}^{\delta}$ is the Schubert structure constant $c_{u,v}^{w}$ for some permutations u, v, w and vice versa.

2.3. **Bumpless pipedreams.** The *(reduced) bumpless pipedreams (BPD)*, introduced by Lam, Lee and Shimozono[LLS21], are combinatorial objects that give a monomial expansion of a Schubert polynomial. For permutation $w \in S_n$, a BPD is an $n \times n$ grid built by the following six tiles:



We adopt the convention that row 1 is the topmost row and column 1 is the left most column. For each $i \in [n]$, we require a pipe to enter from the bottom of column i and end at the rightmost edge of row $w(i)$. Moreover, two pipes cannot cross more than once.

Example 2.1. There are three BPDs for the permutation in S_4 with one-line notation $[2, 1, 4, 3]$.



We let $\mathsf{BPD}(w)$ be the set of BPDs of a permutation w. We call $\square$ a *blank*. The *blank-weight* of a BPD D is a weak composition where the i^{th} entry counts the number of $\square$ in row i. We denote it as $\mathsf{wt}_\square(D)$ to emphasize that the weight comes from the blanks. Then BPD gives a combinatorial formula for Schubert polynomials.

Theorem 2.2 ([LLS21])**.** *For a permutation $w \in S_n$,*

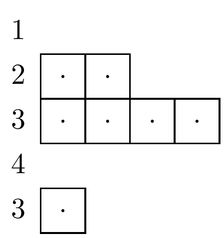
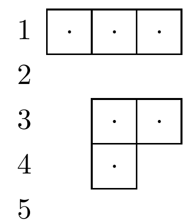
$$\mathfrak{S}_w = \sum_{D \in \mathsf{BPD}(w)} x^{\mathsf{wt}_\square(D)}.$$

For instance, by Example 2.1, when w has one-line notation $[2, 1, 4, 3]$, $\mathfrak{S}_w = x_1x_3 + x_1x_2 + x_1^2$.

2.4. **Diagrams.** A *diagram* is a finite subset of $\mathbb{N} \times \mathbb{N}$. We may represent a diagram by putting a cell at row r and column c for each (r, c) in the diagram. The *weight* of a diagram D, denoted as $\mathsf{wt}(D)$, is a weak composition whose i^{th} entry is the number of boxes in its i^{th} row. Each weak composition α is associated with a diagram $D(\alpha)$, the unique left-justified diagram with weight α. Each permutation $w \in S_n$ or S_+ is associated with a diagram called the *Rothe diagram*

$$RD(w) := \{(r, c) : w(r) > c, w(i) \neq c \text{ for any } i \in [r]\}.$$

Example 2.3. We provide examples of two diagrams. For clarity, we put an "i" on the left of the i^{th} row and put a small dot in each cell.

$D((0, 2, 4, 0, 1)) =$  , $RD([4, 1, 5, 3, 2]) =$  .

The Rothe diagram can characterize one special term in a Schubert polynomial. We consider the *tail-lexicographical order* on weak compositions: For two weak compositions α, γ, we say α is larger than γ if there exists i such that $\alpha_j = \gamma_j$ for all $j > i$ and $\alpha_i > \gamma_i$. For a polynomial f, the *support* of f, denoted as $\mathsf{supp}(f)$, is the set of weak composition α such that x^α has non-zero coefficient in f. The *leading monomial* of f is x^α such that α is the largest in $\mathsf{supp}(\alpha)$. By Lascoux and Schützenberger [LS82a], the leading monomial of $\mathfrak{S}_w$ is $x^{\mathsf{wt}(RD(w))}$ with coefficient 1.

To describe the leading monomial of a top Lascoux polynomial, Pan and Yu [PY23] introduce the *snow diagram*. For each diagram D, its snow diagram $\mathsf{snow}(D)$ is a diagram together with some labels in its cells. Each cell can be unlabeled, or labeled by $\bullet$ or $\circledast$. We only consider the snow diagram of $D(\alpha)$ where α is a snowy weak composition. In this case, $\mathsf{snow}(D(\alpha))$ can be defined as follows. In $D(\alpha)$, label the rightmost cell on each row with $\bullet$. Then put a cell labeled by $\circledast$ in empty spaces above each $\bullet$.

Example 2.4. Let $\alpha = (2, 0, 4, 0, 1)$. Then $D(\alpha)$ and $\mathsf{snow}(D(\alpha))$ are depicted as follows.

$$D(\alpha) = \begin{array}{c|c|c|c|c|} 1 & \cdot & \cdot & \multicolumn{2}{c}{} \\ 2 & \multicolumn{4}{c}{} \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 4 & \multicolumn{4}{c}{} \\ 5 & \cdot & \multicolumn{3}{c}{} \end{array}\ , \qquad \mathsf{snow}(D(\alpha)) = \begin{array}{c|c|c|c|c|} 1 & \cdot & \bullet & & \circledast \\ 2 & \circledast & & & \circledast \\ 3 & \cdot & \cdot & \cdot & \bullet \\ 4 & \circledast & \multicolumn{3}{c}{} \\ 5 & \bullet & \multicolumn{3}{c}{} \end{array}\ .$$

For a snowy α, define

$$\mathsf{rajcode}(\alpha) := \mathsf{wt}(\mathsf{snow}(D(\alpha))). \text{ Equivalently, } \mathsf{rajcode}(\alpha)_i := \alpha_i + \{j > i : \alpha_j > \alpha_i\}.$$

By [PY23], $x^{\mathsf{rajcode}(\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$. Moreover, two distinct snowy weak compositions have different $\mathsf{rajcode}$. We end this subsection with a simple property of $\mathsf{rajcode}$ that will be useful in §4.

Lemma 2.5. *Let α, γ be two snowy weak compositions. If α is larger than γ in tail-lexicographical order, then* $\mathsf{rajcode}(\alpha)$ *is also larger than* $\mathsf{rajcode}(\gamma)$.

Proof. Find i such that $\alpha_j = \gamma_j$ for all $j > i$ and $\alpha_i > \gamma_i$. Clearly, $\mathsf{rajcode}(\alpha)_j = \mathsf{rajcode}(\gamma)_j$ for all $j > i$ and $\mathsf{rajcode}(\alpha)_i > \mathsf{rajcode}(\gamma)_i$. □

3. Relations between top Lascoux polynomials and Schubert polynomials

This section describes the relationship between top Lascoux and Schubert polynomials.

3.1. The reverse complement involution on polynomials.

In this subsection, we describe a linear operator on polynomials. In the next subsection, we use this operator to transform a top Lascoux polynomial into a Schubert polynomial. We begin with an involution on certain weak compositions.

Definition 3.1. Let m, n be positive integers. Define the *reverse complement* operator $r_{m,n}$ on the set of weak compositions α such that $\mathsf{supp}(\alpha) \subseteq [n]$ and $\alpha_i \leqslant m$ for all i. We define

$$r_{m,n}(\alpha) := (m - \alpha_n, \ldots, m - \alpha_1).$$

Next, we analogously define $r_{m,n}$ on certain polynomials.

Definition 3.2. Let m, n be positive integers. We extend $r_{m,n}$ to the set of polynomials in $x_1, \ldots, x_n$ where the power of any x_i is at most m. We define it as the linear operator such that $r_{m,n}(x^\alpha) := x^{r_{m,n}(\alpha)}$. Equivalently, we can define $r_{m,n}$ as

$$r_{m,n}(f) := x_1^m \cdots x_n^m f(x_n^{-1}, \cdots, x_1^{-1}).$$

Remark 3.3. The operator $r_{m,n}$ on polynomials is similar to the operator

$$f \mapsto x_1^n \cdots x_n^n f(x_1^{-1}, \cdots, x_n^{-1})$$

considered by Huh, Matherne, Mészáros and St. Dizier [HMMSD22]. In [HMMSD22, Theorem 6], the authors apply this operator on a Schubert polynomial $\mathfrak{S}_w$ with $w \in S_n$ and show the resulting polynomial is Lorentzian after normalization. Our $r_{m,n}$ is also similar to the operator

$$f \mapsto f(x_1^{-1}, \cdots, x_n^{-1})$$

which sends the character of a GL_n module to the character of its dual. A more generalized operator $f \mapsto x^\alpha f(x_1^{-1}, \cdots, x_n^{-1})$ and its action on Schubert polynomials are studied by Fan, Guo and Liu [FGL20].

Next, we investigate how to swap $r_{m,n}$ with the operators: ∂_i, π_i, and $\widehat{\pi}_i$.

Lemma 3.4. *Suppose $r_{m,n}$ is defined on a polynomial f. Take $i \in [n-1]$. Clearly, $r_{m,n}$ is also defined on $\partial_i(f), \pi_i(f)$, and $\widehat{\pi}_i(f)$. Then we have*

$$\begin{aligned} r_{m,n}(\partial_i(f)) &= \widehat{\pi}_{n-i}(r_{m,n}(f)), \\ r_{m,n}(\pi_i(f)) &= \pi_{n-i}(r_{m,n}(f)), \\ r_{m,n}(\widehat{\pi}_i(f)) &= \partial_{n-i}(r_{m,n}(f)). \end{aligned}$$

Proof. It is enough to assume $f = x^\alpha$, which is a routine check. □

3.2. **Relating Schubert polynomials to top Lascoux polynomials.** In this subsection, we establish that each Schubert polynomial is the reverse complement of a top Lascoux polynomial and vice versa. We start by describing a variation of the map introduced by Fulton [Ful92, (3.4)].

Definition 3.5. Let α be a snowy weak composition. Take any m, n such that $\mathsf{supp}(\alpha) \subseteq [n]$ and $m \geqslant \max(\alpha)$. The *(m,n)-standardization* of α, denoted as $\mathsf{std}_{m,n}(\alpha)$ is the unique permutation w satisfying $w(n+1) < w(n+2) < \cdots$ and

$$w(i) := \begin{cases} r_{m+1,n}(\alpha)_i & \text{if } r_{m+1,n}(\alpha)_i \leqslant m \\ m + |\{j \in [i] : r_{m+1,n}(\alpha)_j = m+1\}| & \text{if } r_{m+1,n}(\alpha)_i = m+1, \end{cases}$$

for any $i \in [n]$.

For instance, if $\alpha = (2,4,0,6,0,0,1)$, then $std_{6,7}(\alpha)$ has one-line notation $[6,7,8,1,9,3,5,2,4]$.
Then we can describe the relation between top Lascoux polynomials and Schubert polynomials.

Theorem 3.6. *Let α be a snowy weak composition. Take any m, n such that $\mathsf{supp}(\alpha) \subseteq [n]$ and $m \geqslant \max(\alpha)$. Let w be the permutation $\mathsf{std}_{m,n}(\alpha)$. Then*

$$r_{m,n}(\widehat{\mathfrak{L}}_\alpha) = \mathfrak{S}_w.$$

For instance, let $\alpha = (2,4,0,6,0,0,1)$, $m = 6$ and $n = 7$. Then $r_{6,7}(\widehat{\mathfrak{L}}_\alpha) = \mathfrak{S}_{[6,7,8,1,9,3,5,2,4]}$.

Proof. Prove by induction on α. For the base case, assume α is a partition with $\mathsf{supp}(\alpha) = [k]$. Then $r_{m+1,n}(\alpha) = (m+1, \cdots, m+1, m+1-\alpha_k, \cdots, m+1-\alpha_1)$. The first n numbers in the one-line notation of w are

$$m+1, m+2, \cdots, m+n-k, m+1-\alpha_k, \cdots, m+1-\alpha_1.$$

Thus, we have $\mathsf{invcode}(w) = (m, \cdots, m, m-\alpha_k, \cdots, m-\alpha_1) = r_{m,n}(\alpha)$, so w is a dominant permutation. By the definition of Schubert polynomials,

$$\mathfrak{S}_w = x^{r_{m,n}(\alpha)} = r_{m,n}(x^\alpha) = \widehat{\mathfrak{L}}_\alpha.$$

Now suppose $\alpha_i < \alpha_{i+1}$. It is routine to check $w(n-i) < w(n-i+1)$ and $\mathsf{std}_{m,n}(s_i\alpha) = ws_{n-i}$. Then the proof is finished by Lemma 3.4:

$$\mathfrak{S}_w = \partial_{n-i}(\mathfrak{S}_{ws_{n-i}}) = \partial_{n-i}(\mathfrak{S}_{\mathsf{std}_{m,n}(s_i\alpha)}) = \partial_{n-i}(r_{m,n}(\widehat{\mathfrak{L}}_{s_i\alpha})) = r_{m,n}(\widehat{\pi}_i(\widehat{\mathfrak{L}}_{s_i\alpha})) = r_{m,n}(\widehat{\mathfrak{L}}_\alpha).$$

□

Example 3.7. We can understand Theorem 3.6 via a commutative diagram. For instance, the equation $r_{4,5}(\widehat{\mathfrak{L}}_{(2,0,4,0,1)}) = \widehat{\mathfrak{L}}_{[4,5,1,6,3,2]}$ is implied by the following commutative diagram.

$$\begin{array}{ccccccccc}
\mathfrak{S}_{[5,6,4,3,1,2]} & \xrightarrow{\partial_2} & \mathfrak{S}_{[5,4,6,3,1,2]} & \xrightarrow{\partial_1} & \mathfrak{S}_{[4,5,6,3,1,2]} & \xrightarrow{\partial_4} & \mathfrak{S}_{[4,5,6,1,3,2]} & \xrightarrow{\partial_3} & \mathfrak{S}_{[4,5,1,6,3,2]} \\
\updownarrow r_{4,5} & & \updownarrow r_{4,5} & & \updownarrow r_{4,5} & & \updownarrow r_{4,5} & & \updownarrow r_{4,5} \\
\widehat{\mathfrak{L}}_{(4,2,1,0,0)} & \xrightarrow{\widehat{\pi}_3} & \widehat{\mathfrak{L}}_{(4,2,0,1,0)} & \xrightarrow{\widehat{\pi}_4} & \widehat{\mathfrak{L}}_{(4,2,0,0,1)} & \xrightarrow{\widehat{\pi}_1} & \widehat{\mathfrak{L}}_{(2,4,0,0,1)} & \xrightarrow{\widehat{\pi}_2} & \widehat{\mathfrak{L}}_{(2,0,4,0,1)}
\end{array}$$

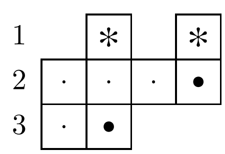
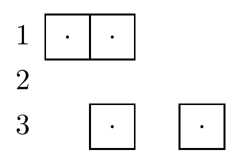
Consequently, every Schubert polynomial is the reverse complement of a top Lascoux polynomial.

Corollary 3.8. *Consider $w \in S_n$. Let $\alpha = (n+1-w(n), \cdots, n+1-w(2), n+1-w(1))$. Then $\mathfrak{S}_w = r_{n,n}(\widehat{\mathfrak{L}}_\alpha)$.*

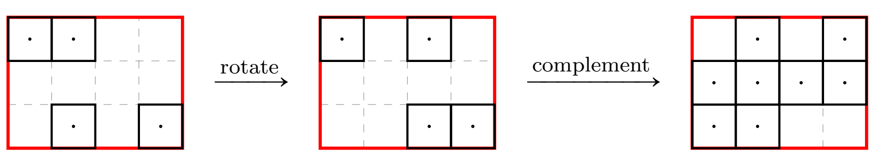
Proof. Notice that $\mathsf{std}_{n,n}(\alpha) = w$. Then the proof is finished by Theorem 3.6. □

3.3. **A diagrammatic perspective of standardization.** In this subsection, we interpret the standardization map $\mathsf{std}_{m,n}$ in terms of diagrams. Recall that each snowy weak composition α is associated with a labeled diagram $\mathsf{snow}(D(\alpha))$. Each permutation w is associated with the Rothe diagram $RD(w)$. We describe the relationship between $\mathsf{snow}(D(\alpha))$ and $RD(\mathsf{std}_{m,n}(\alpha))$.

Example 3.9. Consider the snowy weak composition $\alpha = (0,4,2)$. Let $m = 4$ and $n = 3$. Let $w = \mathsf{std}_{m,n}(\alpha) = [3,1,5,2,4]$. We depict $\mathsf{snow}(D(\alpha))$ and $RD(w)$ as follows:

$\mathsf{snow}(D(\alpha)) =$  $RD(w) =$ 

We observe that both diagrams live in the first n rows and m columns. Imagine that we put $RD(w)$ in an $n \times m$ box and rotate it by 180°. Then take its complement within the $n \times m$ box. What we get is exactly $\mathsf{snow}(D(\alpha))$ without labels.



This observation holds in general.

Lemma 3.10. *Let α be a snowy weak composition. Take n with $\mathsf{supp}(\alpha) \subseteq [n]$ and m with $\max(\alpha) \leqslant m$. Let $w = \mathsf{std}_{m,n}(\alpha)$. Then both $\mathsf{snow}(D(\alpha))$ and $RD(w)$ live in the first n rows and m columns. If we embed $RD(w)$ in an $n \times m$ box, rotate it by 180° and complement it from the $n \times m$ box, what we get is $\mathsf{snow}(D(\alpha))$ without labels.*

Proof. If we ignore the labels, $\mathsf{snow}(D(\alpha))$ consists of cells (r,c) such that $\alpha_r \geqslant c$ or $\alpha_i = c$ for some $i > r$. Clearly, $\mathsf{snow}(D(\alpha))$ lies in the first n rows and m columns. If we complement $\mathsf{snow}(D(\alpha))$ from the $n \times m$ box, the resulting diagram consists of $(r,c) \in [m] \times [n]$ such that $\alpha_r < c$ and $\alpha_i \neq c$ for any $i > r$. Now we put this diagram in the $n \times m$ box and rotate it by 180°. Let D' be the resulting diagram. We check D' and $RD(w)$ agree row by row. They clearly agree under row n. Consider $r \in [n]$. If $\alpha_{n+1-r} = 0$, then D' has no cells in row r. Also, $w(r) \geqslant m+1$. By the definition of standardization map, there are no inversions of w of the form (r, r'). Thus, $RD(w)$ also has no cells in row r. If $\alpha_{n+1-r} > 0$, D' has cells in column c for c such that $c < m+1-\alpha_{n+1-r}$ and $c \neq m+1-\alpha_{n+1-i}$ for all $i \in [r]$. In other words, that is all c such that $c < w(r)$ and c is not in $w(1), \cdots, w(r)$. Clearly, D' and $RD(w)$ agree on row r.

□

4. Relations between top Lascoux structure constants and the Schubert structure constants

Recall that $\{\widehat{\mathfrak{L}}_\alpha : \alpha \text{ is snowy.}\}$ form a basis of the algebra $\widehat{V}$. The top Lascoux structure constant $d^\delta_{\alpha,\gamma}$ is the coefficient of $\widehat{\mathfrak{L}}_\delta$ in the expansion of $\widehat{\mathfrak{L}}_\alpha \widehat{\mathfrak{L}}_\gamma$. At this point, we do not have any reason to believe that they are positive integers. Surprisingly, the connection between top Lascoux polynomials and Schubert polynomials establishes a bridge between $d^\delta_{\alpha,\gamma}$ and the Schubert structure constants $c^w_{u,v}$. First, we describe a necessary condition for $d^\delta_{\alpha,\gamma}$ to be non-zero.

Lemma 4.1. *Let α, γ and δ be snowy weak compositions. Find m_1, m_2 and n such that $m_1 \geqslant \max(\alpha)$, $m_2 \geqslant \max(\gamma)$, $\mathsf{supp}(\alpha) \subseteq [n]$ and $\mathsf{supp}(\gamma) \subseteq [n]$. If $d^\delta_{\alpha,\gamma} \neq 0$, we must have $\mathsf{supp}(\delta) \subseteq [n]$ and $\max(\delta) \leqslant m_1 + m_2$.*

The proof relies heavily on the statistic $\mathsf{rajcode}$.

Proof. First, expand

$$\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma = \sum_\sigma d^\sigma_{\alpha,\gamma} \widehat{\mathfrak{L}}_\sigma. \tag{3}$$

We know the left hand side uses only variables $x_1, \ldots, x_n$. Moreover, in any monomial on the left hand side, each variable has power at most $m_1 + m_2$. Let S be the set of all σ with $d^\sigma_{\alpha,\gamma} \neq 0$. Among S, find the largest σ in tail-lexicographical order. By Lemma 2.5, $\mathsf{rajcode}(\sigma)$ is also larger than $\mathsf{rajcode}(\sigma')$ for any $\sigma' \in S$. Thus, $x^{\mathsf{rajcode}(\sigma)}$ has non-zero coefficient on the right hand side of (3), so $\mathsf{supp}(\sigma) \subseteq [n]$. It follows that $\mathsf{supp}(\sigma') \subseteq [n]$ for any $\sigma' \in S$.

Now find $\sigma \in S$ with the largest $\max(\sigma)$, break ties by picking the largest in tail-lexicographical order. Say $\max(\sigma) = m$. In $\mathsf{rajcode}(\alpha)$, one entry is m. We can see $\mathsf{rajcode}(\alpha)$ cannot appear in $\widehat{\mathfrak{L}}_{\sigma'}$ for any other $\sigma' \in S$: If so, then σ' has an entry at least m and $\mathsf{rajcode}(\sigma')$ is larger than $\mathsf{rajcode}(\sigma)$, contradicting to the maximality of σ. Thus, $x^{\mathsf{rajcode}}(\sigma)$ has non-zero coefficient on the right hand side of (3), so $m \leqslant m_1 + m_2$. It follows that $\max(\sigma') \leqslant m_1 + m_2$ for any $\sigma' \in S$. □

Now we describe the main theorem of this section.

Theorem 4.2. *Let α, γ be snowy weak compositions. Find m_1, m_2 and n such that $m_1 \geqslant \max(\alpha)$, $m_2 \geqslant \max(\gamma)$, $\mathsf{supp}(\alpha) \subseteq [n]$ and $\mathsf{supp}(\gamma) \subseteq [n]$. Let $u = \mathsf{std}_{m_1,n}(\alpha)$ and $v = \mathsf{std}_{m_2,n}(\gamma)$. For any snowy weak composition δ with $\mathsf{supp}(\delta) \subseteq [n]$ and $\max(\delta) \leqslant m_1 + m_2$, we let $w = \mathsf{std}_{m_1+m_2,n}(\delta)$. Then $d^\delta_{\alpha,\gamma} = c^w_{u,v}$.*

Proof. First, we have $\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma = \sum_\sigma d^\sigma_{\alpha,\gamma} \widehat{\mathfrak{L}}_\sigma$. By Lemma 4.1, the sum is over all σ with $\mathsf{supp}(\sigma) \subseteq [n]$ and $\max(\sigma) \leqslant m_1 + m_2$. Apply $r_{m_1+m_2,n}$ on both sides. Using Theorem 3.6, the left hand side becomes

$$r_{m_1+m_2,n}(\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\gamma) = r_{m_1,n}(\widehat{\mathfrak{L}}_\alpha) r_{m_2,n}(\widehat{\mathfrak{L}}_\gamma) = \mathfrak{S}_u \mathfrak{S}_v.$$

The right hand becomes

$$r_{m_1+m_2,n}(\sum_\sigma d^\sigma_{\alpha,\gamma} \widehat{\mathfrak{L}}_\sigma) = \sum_\sigma d^\sigma_{\alpha,\gamma} r_{m_1+m_2,n}(\widehat{\mathfrak{L}}_\sigma) = \sum_\sigma d^\sigma_{\alpha,\gamma} \mathfrak{S}_{\mathsf{std}_{m_1+m_2,n}(\sigma)}.$$

□

Example 4.3. Let $\alpha = (2,3,1,4)$ and $\gamma = (2,1,4,3)$. We can let $m_1 = m_2 = n = 4$. Then $u = \mathsf{std}_{4,4}(\alpha) = [1,4,2,3]$ and $v = \mathsf{std}_{4,4}(\gamma) = [2,1,4,3]$. We compute

$$\widehat{\mathfrak{L}}_\alpha * \widehat{\mathfrak{L}}_\gamma = \widehat{\mathfrak{L}}_{(8,6,5,7)} + \widehat{\mathfrak{L}}_{(6,8,4,7)} + \widehat{\mathfrak{L}}_{(7,8,5,6)} + \widehat{\mathfrak{L}}_{(7,6,8,5)} + \widehat{\mathfrak{L}}_{(6,7,8,4)}$$
$$\mathfrak{S}_u * \mathfrak{S}_v = \mathfrak{S}_{[2,4,3,1]} + \mathfrak{S}_{[2,5,1,3,4]} + \mathfrak{S}_{[3,4,1,2]} + \mathfrak{S}_{[4,1,3,2]} + \mathfrak{S}_{[5,1,2,3,4]}.$$

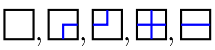
We check Theorem 4.2 when $\delta = (8,6,5,7)$. We have $d^\delta_{\alpha,\gamma} = 1$. Now we compute $w = \mathsf{std}_{4+4,4}(\delta) = [2,4,3,1]$. Indeed, $c^w_{u,v} = 1$.

Theorem 4.2 can express each Schubert structure constant as a top Lascoux structure constant.

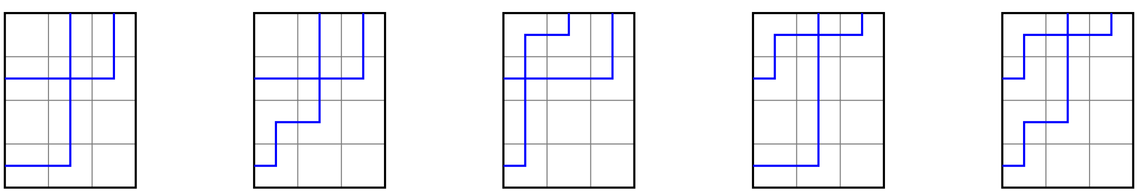
Corollary 4.4. *Take $u, v \in S_n$ and $w \in S_{2n}$. Assume $w(n+1) < \cdots < w(2n)$. Let $\alpha = (n+1-u(n), \cdots, n+1-u(1))$, $\gamma = (n+1-v(n), \cdots, n+1-v(1))$ and $\delta = (n+1-w(n), \cdots, n+1-w(1))$ Then $c^w_{u,v} = d^\delta_{\alpha,\gamma}$.*

5. Bumpless pipedream formula

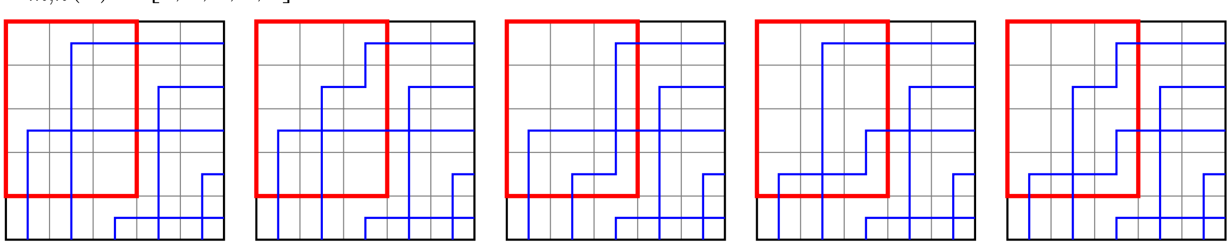
In this section, we discuss another application of the relationship between top Lascoux polynomials and Schubert polynomials. Bumpless pipedreams (BPD) give a formula to compute the monomial expansion of Schubert polynomials. After "reversing the BPDs", we get a formula for top Lascoux polynomials.

Definition 5.1. Let α be an snowy weak composition. Find smallest n such that $\mathsf{supp}(\alpha) \subseteq [n]$ and let $m = \max(\alpha)$. A *left-to-top BPD* of α is a grid with n rows and m columns built by tiles  and . For each $i \in [n]$ with $\alpha_i > 0$, there is a pipe entering from the left in row i and goes to the top of column α_i. Moreover, no two pipes can cross more than once. Let $\mathsf{LTBPD}(\alpha)$ be the set of all left-to-top BPDs of α.

Example 5.2. Consider $\alpha = (0,3,0,2)$. Then $n = 4$ and $m = 3$. The set $\mathsf{LTBPD}(\alpha)$ has five elements:



Remark 5.3. Readers might notice that left-to-top BPDs look like the top left part of a BPD after rotation. Keep α, m and n from Example 5.2. Consider the classical BPDs of the permutation $\mathsf{std}_{m,n}(\alpha) = [2,4,1,5,3]$. There are five of them:



The top-to-left BPDs in Example 5.2 are obtained by rotating the red part of these BPDs.

This pattern holds in general.

Proposition 5.4. *Let α be a snowy weak composition. Find smallest n such that $\mathsf{supp}(\alpha) \in [n]$ and let $m = \max(\alpha)$. Let $w = \mathsf{std}_{m,n}(\alpha)$. Then $\mathsf{LTBPD}(\alpha)$ is formed by rotating the first n rows, m columns of BPDs in $\mathsf{BPD}(w)$.*

Proof. Immediate from Lemma 3.10. ☐

Now we are ready to introduce a combinatorial formula for the top Lascoux polynomials. We just need to specify the "weight" of a left-to-top BPD.

Definition 5.5. Let D be a left-to-top BPD for some snowy weak composition. The *non-blank weight* of D, denoted as $\mathsf{wt}_{\boxdot}(D)$, is a weak composition where the i^{th} entry counts the number of **non-blank tiles** in row i of D.

Theorem 5.6. *Let α be a snowy weak composition. Then*

$$\widehat{\mathfrak{L}}_\alpha = \sum_{D \in \mathsf{LTBPD}(\alpha)} \mathsf{wt}_{\boxdot}(D).$$

For instance, Example 5.2 yields

$$\widehat{\mathfrak{L}}_{(0,3,0,2)} = x_1^2x_2^3x_3x_4^2 + x_1^2x_2^3x_3^2x_4 + x_1^3x_2^3x_3x_4 + x_1^3x_2^2x_3x_4^2 + x_1^3x_2^2x_3^2x_4.$$

Proof. Follows from Theorem 3.6 and Proposition 5.4. ☐

Remark 5.7. One can also prove Theorem 5.6 by an induction on α. The key step is to show

$$\widehat{\pi}_i \left(\sum_{D \in LTBPD(s_i\alpha)} \mathsf{wt}_{\boxdot}(D) \right) = \sum_{D \in LTBPD(\alpha)} \mathsf{wt}_{\boxdot}(D)$$

for α with $\alpha_i < \alpha_{i+1}$. This argument can be formed by slightly modifying the proof of [Hua21, Proposition 2.1].

6. Support of top Lascoux polynomials

In this section, we study the support of top Lascoux polynomials. Our main tool is an elegant description of the support of Schubert polynomials given by Adve, Robichaux, and Yong [ARY21].

Definition 6.1. [ARY21] Let D be a diagram and α be a weak composition. Then $\mathsf{PerfectTab}_\downarrow(D, \alpha)$ is the set of fillings of D satisfying all of the following:

- For each k, the number of cells in D filled by k is α_k.
- In each column, numbers are increasing from top to bottom.
- Any entry in row i is at most i.

Example 6.2. [ARY21, Example 1.4] Consider $D = RD((3,1,5,2,4))$. We enumerate the six elements in $\bigcup_\alpha \mathsf{PerfectTab}_\downarrow(D, \alpha)$:

1	1	1		
2				
3		2		1

1	1	1		
2				
3		2		2

1	1	1		
2				
3		2		3

1	1	1		
2				
3		3		1

1	1	1		
2				
3		3		2

1	1	1		
2				
3		3		3

Theorem 6.3. [ARY21, Theorem 1.3] *For a permutation w, the support of $\mathfrak{S}_w$ is the set of α such that $\mathsf{PerfectTab}_\downarrow(RD(w), \alpha) \neq \varnothing$.*

For instance, Example 6.2 says that $\mathsf{supp}(\mathfrak{S}_{(3,1,5,2,4)})$ consists of $(3,1), (2,2), (2,1,1), (3,0,1)$ and $(2,0,2)$.

The support of a top Lascoux polynomial can be characterized in the same manner.

Proposition 6.4. *For a snowy weak composition α, let D be the diagram $\mathsf{snow}(D(\alpha))$ with labels erased. The support of $\widehat{\mathfrak{L}}_\alpha$ is the set of γ such that $\mathsf{PerfectTab}_\downarrow(D,\gamma) \neq \varnothing$.*

Example 6.5. Consider the snowy weak composition $(0,4,2)$. Ignore all labels in $\mathsf{snow}(D(\alpha))$ and obtain the diagram D. We enumerate the six elements in $\bigcup_\alpha \mathsf{PerfectTab}_\downarrow(D,\alpha)$:

1		1		1
2	1	2	1	2
3	2	3		

1		1		1
2	1	2	1	2
3	3	3		

1		1		1
2	2	2	1	2
3	3	3		

1		1		1
2	1	2	2	2
3	2	3		

1		1		1
2	1	2	2	2
3	3	3		

1		1		1
2	2	2	2	2
3	3	3		

By Proposition 6.4, the support of $\widehat{\mathfrak{L}}_{(0,4,2)}$ consists of $(4,3,1), (4,2,2), (3,3,2), (3,4,1)$ and $(2,4,2)$.

Proof. Find m,n such that $m \geqslant \max(\alpha)$ and $\mathsf{supp}(\alpha) \subseteq [n]$. Let $w = \mathsf{std}_{m,n}(\alpha)$. By Theorem 3.6, we have $\mathsf{supp}(\widehat{\mathfrak{L}}_\alpha) = \{r_{m,n}(\gamma) : \gamma \in \mathsf{supp}(\mathfrak{S}_w)\}$. It remains to build a bijection from $\mathsf{PerfectTab}_\downarrow(RD(w),\gamma)$ to $\mathsf{PerfectTab}_\downarrow(\mathsf{snow}(D(\alpha)), r_{m,n}(\gamma))$.

We take T from $\mathsf{PerfectTab}_\downarrow(RD(w),\gamma)$. By Lemma 3.10, we may embed T in an n by m box. We rotate it by 180° and replace each number i in T by $n+1-i$. Now we have a filling of some diagram where each column is increasing from top to bottom and any entry in row i is at least i. By Lemma 3.10, for each c in $[m]$, if column c of the current T has cells in row $r_1,\dots,r_k$, then column c of $\mathsf{snow}(D(\alpha))$ has cells in row $[n]-\{r_1,\dots,r_k\}$. We fill column c of $\mathsf{snow}(D(\alpha))$ with numbers in $[n]$ that are not in column c of T. It is routine to check this is a well-defined bijection. □

Example 6.6. We apply the bijection to one element shown in Example 6.2:

1	1	1		
2				
3		2		3

$\xrightarrow{\text{rotate}}$

1	3		2	
2				
3			1	1

$\xrightarrow{\text{replace } i \text{ by } 4-i}$

1	1		2	
2				
3			3	3

Then we fill column 1 of $\mathsf{snow}(D(\alpha))$ with $[3]-\{1\} = \{2,3\}$, fill column 2 with $[3]-\varnothing = [3]$, fill column 3 with $[3]-\{2,3\} = \{1\}$ and fill column 4 with $[3]-\{3\} = \{1,2\}$. We get the third element in Example 6.5.

The support of a Schubert polynomial has one nice property.

Definition 6.7. [MTY19] Consider $f \in \mathbb{Q}[x_1,\cdots]$ with $\mathsf{supp}(f) \subseteq [n]$. We may view $\mathsf{supp}(f)$ as a subset of $\mathbb{R}^n$ and consider its convex hull. If any lattice point in the convex hull is also in $\mathsf{supp}(f)$, we say f has *saturated Newton polytope* (SNP).

Fink, Mészáros, and St. Dizier show that Schubert polynomials has SNP [FMSD18, Corollary 8]. We move this property to top Lascoux polynomials.

Proposition 6.8. *For any snowy α, $\widehat{\mathfrak{L}}_\alpha$ has SNP.*

Proof. By Theorem 3.6, $\widehat{\mathfrak{L}}_\alpha = r_{m,n}(\mathfrak{S}_w)$ for some m,n,w. The SNP property is clearly preserved by $r_{m,n}(\cdot)$. □

7. Key expansion of top Lascoux polynomials

In this section, we expand top Lascoux polynomials positively into key polynomials. Our main tool is the Schubert-to-key expansion:

Theorem 7.1 ([RS95])**.** *For $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ can be expanded as $\sum_\alpha c_\alpha^w \kappa_\alpha$, where the coefficients c_α^w are non-negative integers counting certain tableaux.*

We will translate this expansion to top Lascoux polynomials. The first step is to understand how the $r_{m,n}$ operator affects a key polynomial.

Proposition 7.2. *Let α be any weak composition. Let m, n be positive integers. Then $r_{m,n}$ is defined on α if and only if $r_{m,n}$ is defined on κ_α. If this is the case, $r_{m,n}(\kappa_\alpha) = \kappa_{r_{m,n}(\alpha)}$.*

The proof is essentially the same as the proof of Theorem 3.6.

Proof. The first claim is immediate. We prove the equation by induction on α. For the base case, assume α is a partition. So is $r_{m,n}(\alpha)$. We have $r_{m,n}(\kappa_\alpha) = r_{m,n}(x^\alpha) = x^{r_{m,n}(\alpha)} = \kappa_{r_{m,n}(\alpha)}$.

Now assume $\alpha_i < \alpha_{i+1}$ for some i. For our inductive hypothesis, assume $\kappa_{s_i\alpha} = r_{m,n}(\kappa_{r_{m,n}(s_i\alpha)})$. By Lemma 3.4,

$$\kappa_\alpha = \pi_i(\kappa_{s_i\alpha}) = \pi_i(r_{m,n}(\kappa_{r_{m,n}(s_i\alpha)})) = r_{m,n}(\pi_{n-i}(\kappa_{s_{n-i}(r_{m,n}(\alpha))})) = r_{m,n}(\kappa_{r_{m,n}(\alpha)}).$$

□

Corollary 7.3. *Let α be a snowy weak composition. Find m, n such that $\alpha \subseteq [n]$ and $\max(\alpha) \leqslant m$. Let w be the permutation $\mathsf{std}_{m,n}(\alpha)$. The top Lascoux polynomial $\widehat{\mathfrak{L}}_\alpha$ can be expanded as $\sum_\gamma c_\gamma^w \kappa_{r_{m,n}(\gamma)}$ where the coefficient c_γ^w is the coefficient of κ_γ in the expansion of $\mathfrak{S}_w$.*

Proof. By Theorem 3.6, we know $\widehat{\mathfrak{L}}_\alpha = r_{m,n}(\mathfrak{S}_w)$. By the Schubert-to-key expansion and Proposition 7.2,

$$\widehat{\mathfrak{L}}_\alpha = r_{m,n}(\sum_\gamma c_\gamma^w \kappa_\gamma) = \sum_\gamma c_\gamma^w r_{m,n}(\kappa_\gamma) = \sum_\gamma c_\gamma^w \kappa_{r_{m,n}(\gamma)}.$$

□

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(T. Yu) Department of Mathematics, UC San Diego, La Jolla, CA 92093, U.S.A.
Email address: `tiy059@ucsd.edu`